APPENDIX

Computer Code for Computing Transition Probabilities with Initial and Final Backward Times

Symbol $*$ means the element-by-element or Hadamard matrix product. Symbol $\bullet$ means the usual row-column matrix product. The names of the variables written in italics are real numbers; the ones written in boldface are matrices or vectors, i.e. $p_{ij}(t)$ is an element of the matrix $\mathbf{P}(t)$.

Inputs:
T: time horizon considered
m: number of states
$\mathbf{P}(t)$: matrix of the non-homogeneous embedded Markov chain
$\mathbf{G}(s,t)$: matrix of the waiting time distribution functions

(* kernel construction*)
FOR $s = 0,\ s \le T,\ s++,$
  FOR $t = s+1,\ t \le T,\ t++,$
    $\mathbf{Q}(s,t) = \mathbf{P}(s) * \mathbf{G}(s,t);$
  END FOR;
END FOR;

(* $H_i(s;t)$: probability to exit from state i *)
FOR $s = 0,\ s \le T,\ s++,$
  FOR $t = s+1,\ t \le T,\ t++,$
    FOR $i = 1, i \le m, i++,$
      $H_i(s;t) = \sum_{j=1}^{m} Q_{ij}(s;t);$
    END FOR;
  END FOR;
END FOR;

(* $d_{ii}(v,s;t)$: probability to have no transition from state i with initial backward time $s - v$ *)
FOR $v = 0, v \le T, v++,$
  FOR $s = v,\ s \le T,\ s++,$
    FOR $t = s+1,\ t \le T,\ t++,$
      FOR $i = 1, i \le m, i++,$
        $d_{ii}(v,s;t) = \left(1 - H_i(v,t)\right) / \left(1 - H_i(v,s)\right);$
      END FOR;
    END FOR;
  END FOR;
END FOR;

(* $\mathbf{B}(v,s;t)$ probability to go from state i to state j just at time t with initial backward time s-u *)

FOR $v=0, v\le T, v++,$

FOR $s=v,\ s\le T,\ s++,$

FOR $t=s+1,\ t\le T,\ t++,$

FOR $i=1, i\le m, i++,$

FOR $j=1,\ j\le m,\ j++,$

$$b_{ij}(v,s;t)=\left(Q_{ij}(v,t)-Q_{ij}(v,t-1)\right)\Big/\left(1-H_i(v,s)\right);$$

END FOR;

END FOR;

END FOR;

END FOR;

END FOR;

(* solution of NHSMP initial and final backward evolution equation *)

(* step 1 – $\mathbf{\Phi}^b(l,l;l,t)$ transition probabilities with final backward *)

FOR $t=T,\ t>0,\ t--,$

FOR $l=t, l\ge 0, l--,$

FOR $i=1, i\le m, i++,$

$\phi_{ii}^b(l,l;l,t)=d_{ii}(l,l;t)$; (*probability to remain always in i; if $i\ne j,\ d_{ij}=0$ *)

END FOR;

FOR $s=l-1, s\ge 0, s--,$

FOR $k=l, k>s, k--,$

$\mathbf{\Phi}^b(s,s;l,t)+=\mathbf{B}(s,s;k)\bullet\mathbf{\Phi}^b(k,k;l,t)$;

END FOR;

END FOR;

END FOR;

END FOR;

(* step 2 – ${}^b\mathbf{\Phi}^b(v,s;v',t)$ transition probabilities with initial and final backward *)

FOR $t=T, t\ge 1, t--,$

FOR $l=t, l\ge 0,\ l--,$

FOR $v=l-1,\ v\ge 0,\ v--,$

FOR $i=1, i\le m, i++,$

${}^b\phi_{ii}^b(v,l;l,t)=d_{ii}(v,l;t)$;

END FOR;

END FOR;

FOR $s=l-1,\ s>0,\ s--,$

FOR $v=s-1, v\ge 0,\ v--,$

FOR $k=l, k>s, k--,$

${}^b\mathbf{\Phi}^b(v,s;l,t)+=\mathbf{B}(v,s;k)\bullet{}^b\mathbf{\Phi}^b(k,k;l,t)$;(*calculation of formula (3)*)

END FOR;

END FOR;

END FOR;

END FOR;

END FOR;

Supplementary Results of Application

Table 1 Number of Instruments by S&P Ratings per Year

	AAA	AA	A	BBB	BB	B	CCC-C	D	NR
1981	182	431	565	424	122	233	30	4	228
1982	383	632	708	538	141	101	57	36	317
1983	1194	835	818	533	241	173	61	14	354
1984	788	1525	1046	603	297	198	96	23	410
1985	1507	1659	1947	1102	420	413	178	39	710
1986	1184	1932	1826	1581	521	585	385	106	1622
1987	1002	1918	1809	1084	462	482	226	82	2227
1988	1215	1489	1962	1080	358	408	242	110	1818
1989	848	1703	1836	1270	434	419	208	61	1996
1990	1078	2049	2925	1392	336	223	242	141	2245
1991	1121	2425	3206	1483	507	252	231	171	2904
1992	1289	2169	3088	1652	468	532	120	70	3865
1993	1380	1832	3155	1791	824	649	108	26	4653
1994	2387	1414	2893	1492	483	413	97	23	3298
1995	1303	1986	3227	1732	554	478	85	52	2723
1996	2162	2113	2739	1910	753	718	92	26	4430
1997	2310	2577	3612	2280	1509	1228	137	34	4681
1998	3465	3086	3764	2813	1514	1355	336	65	5264
1999	2938	3347	4408	3363	1499	1528	505	226	7743
2000	5116	4335	6127	3522	1575	1270	615	304	8305
2001	4117	4148	6479	5343	2023	1814	1382	603	9585
2002	7411	4345	8407	6317	2798	2342	1470	888	11919
2003	11570	5822	8465	6757	2929	2930	1026	547	15195
2004	9307	7906	7502	6227	3123	2746	726	127	16172
2005	10193	9148	10655	7717	5931	2796	768	238	16517
2006	13696	10374	10743	5887	5600	3435	798	135	17811
2007	7889	9717	5969	4467	3284	2173	672	105	11766
2008	13154	14600	16427	9225	5113	5770	6639	2306	23817

Table 2 Embedded Markov Chain at $s = 0$

	AAA	AA	A	BBB	BB	B	CCC	NR1	NR2	D
AAA	0.106	0.127	0.021	0.000	0.000	0.000	0.000	0.746	0.000	0.000
AA	0.034	0.356	0.231	0.017	0.000	0.001	0.000	0.360	0.000	0.000
A	0.007	0.042	0.358	0.183	0.003	0.000	0.000	0.408	0.000	0.000
BBB	0.002	0.008	0.107	0.421	0.142	0.008	0.001	0.312	0.000	0.000
BB	0.000	0.000	0.009	0.302	0.227	0.130	0.006	0.000	0.326	0.000
B	0.000	0.000	0.007	0.012	0.261	0.305	0.066	0.000	0.347	0.002
CCC	0.000	0.000	0.000	0.000	0.000	0.105	0.210	0.000	0.343	0.343
NR1	0.000	0.000	0.013	0.000	0.000	0.000	0.000	0.987	0.000	0.000
NR2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000
D	0.000	0.000	0.000	0.000	0.000	0.023	0.068	0.000	0.909	0.000

Table 3 Semi-Markov Transition Probabilities with Initial Backward for Different Time Values

$^{b}\phi_{ij}$ (2;3,7)

	AAA	AA	A	BBB	BB	B	CCC	NR1	NR2	D
AAA	0.564	0.106	0.016	0.012	0.001	0.000	0.000	0.301	0.000	0.000
AA	0.014	0.500	0.125	0.035	0.003	0.000	0.000	0.324	0.000	0.000
A	0.011	0.063	0.536	0.082	0.009	0.001	0.000	0.297	0.002	0.000
BBB	0.008	0.017	0.163	0.500	0.035	0.004	0.001	0.264	0.009	0.000
BB	0.002	0.004	0.044	0.227	0.353	0.075	0.013	0.068	0.212	0.001
B	0.002	0.002	0.018	0.062	0.191	0.284	0.050	0.015	0.364	0.011
CCC	0.000	0.000	0.003	0.034	0.084	0.138	0.133	0.004	0.584	0.019
NR1	0.017	0.011	0.041	0.087	0.034	0.006	0.001	0.782	0.021	0.000
NR2	0.002	0.001	0.003	0.018	0.085	0.314	0.037	0.001	0.534	0.006
D	0.001	0.001	0.007	0.072	0.132	0.029	0.002	0.017	0.739	0.000

$^{b}\phi_{ij}$ (5;6,10)

	AAA	AA	A	BBB	BB	B	CCC	NR1	NR2	D
AAA	0.315	0.166	0.051	0.022	0.004	0.001	0.000	0.441	0.000	0.000
AA	0.069	0.389	0.106	0.035	0.004	0.001	0.000	0.395	0.000	0.000
A	0.035	0.077	0.395	0.162	0.020	0.004	0.001	0.303	0.003	0.000
BBB	0.035	0.048	0.131	0.407	0.056	0.016	0.003	0.288	0.015	0.001
BB	0.008	0.012	0.029	0.104	0.299	0.166	0.060	0.036	0.267	0.020
B	0.005	0.006	0.012	0.037	0.111	0.313	0.139	0.014	0.318	0.044
CCC	0.002	0.004	0.004	0.018	0.089	0.141	0.285	0.003	0.349	0.103
NR1	0.031	0.028	0.023	0.025	0.006	0.001	0.000	0.876	0.002	0.008
NR2	0.001	0.004	0.002	0.007	0.028	0.116	0.045	0.002	0.788	0.008
D	0.006	0.017	0.010	0.033	0.124	0.102	0.040	0.010	0.529	0.129

$^{b}\phi_{ij}$ (10;11,15)

	AAA	AA	A	BBB	BB	B	CCC	NR1	NR2	D
AAA	0.249	0.119	0.051	0.046	0.024	0.011	0.003	0.494	0.004	0.000
AA	0.071	0.361	0.098	0.040	0.020	0.009	0.002	0.395	0.003	0.000
A	0.039	0.090	0.367	0.091	0.030	0.010	0.002	0.366	0.005	0.000
BBB	0.029	0.028	0.104	0.343	0.107	0.030	0.007	0.309	0.042	0.000
BB	0.005	0.005	0.018	0.119	0.267	0.136	0.067	0.042	0.337	0.005
B	0.001	0.001	0.010	0.037	0.138	0.258	0.131	0.007	0.401	0.015
CCC	0.001	0.001	0.007	0.034	0.096	0.187	0.164	0.005	0.479	0.025
NR1	0.053	0.056	0.081	0.099	0.039	0.017	0.005	0.638	0.013	0.000
NR2	0.001	0.001	0.008	0.084	0.143	0.210	0.115	0.008	0.418	0.012
D	0.004	0.008	0.008	0.038	0.114	0.152	0.123	0.011	0.488	0.055

$^{b}\phi_{ij}$ (9;11,15)

	AAA	AA	A	BBB	BB	B	CCC	NR1	NR2	D
AAA	0.313	0.096	0.049	0.045	0.024	0.012	0.003	0.456	0.003	0.000
AA	0.082	0.358	0.106	0.039	0.020	0.010	0.002	0.380	0.002	0.000
A	0.042	0.083	0.375	0.099	0.031	0.011	0.003	0.352	0.005	0.000
BBB	0.036	0.033	0.089	0.347	0.080	0.024	0.007	0.352	0.030	0.000
BB	0.003	0.003	0.023	0.084	0.307	0.135	0.066	0.034	0.342	0.005
B	0.001	0.001	0.009	0.046	0.135	0.238	0.114	0.012	0.432	0.011
CCC	0.000	0.000	0.007	0.028	0.090	0.196	0.216	0.003	0.433	0.026
NR1	0.075	0.068	0.081	0.063	0.032	0.014	0.004	0.658	0.006	0.000
NR2	0.000	0.000	0.006	0.031	0.108	0.213	0.193	0.003	0.421	0.025
D	0.001	0.000	0.004	0.018	0.065	0.099	0.055	0.002	0.251	0.504

Note: $^{b}\phi_{ij}$ (v;s,t) stands for the semi-Markov transition probabilities with initial backward.

Table 4 Reliability with Initial and Final Backward from Rating AAA and BB

	${}^bR^b_{AAA}$ (4,8;l´,15) ${}^bR^b_{BB}$ (4,8;l´,15)	${}^bR^b_{AAA}$ (5,8;l´,15) ${}^bR^b_{BB}$ (5,8;l´,15)	${}^bR^b_{AAA}$ (6,8;l´,15) ${}^bR^b_{BB}$ (6,8;l´,15)	${}^bR^b_{AAA}$ (7,8;l´,15) ${}^bR^b_{BB}$ (7,8;l´,15)	${}^bR^b_{AAA}$ (8,8;l´,15) ${}^bR^b_{BB}$ (8,8;l´,15)
v' = 8	0,070	0.063	0.145	0.120	0.114
	0.000	0.000	0.008	0.007	0.010
v' = 9	0.017	0.010	0.021	0.016	0.021
	0.008	0.009	0.005	0.008	0.008
v' = 10	0.021	0.027	0.017	0.018	0.026
	0.005	0.006	0.013	0.007	0.008
v' = 11	0.011	0.014	0.012	0.011	0.016
	0.016	0.013	0.014	0.013	0.014
v' = 12	0.027	0.029	0.027	0.027	0.031
	0.018	0.019	0.024	0.021	0.021
v' = 13	0.043	0.047	0.047	0.046	0.047
	0.039	0.038	0.041	0.039	0.038
v' = 14	0.110	0.096	0.100	0.101	0.107
	0.059	0.063	0.060	0.059	0.057
v' = 15	0.700	0.714	0.632	0.657	0.638
	0.179	0.189	0.211	0.178	0.194

Note: ${}^bR^b_i$ ($v,s;\ v',\ t$) stands for the semi-Markov reliabilities with initial and final backward starting from rating class *i*.

Table 5 Maintainability Functions with Initial Backward

	$t = s+1$	$t = s+2$	$t = s+3$	$t = s+4$	$t = s+5$	$t = s+6$	$t = s+7$	$t = s+8$	$t = s+9$	$t = s+10$	$t = s+11$
	${}^bM_{ij}$ (8;8,t)										
NR2	0.356	0.495	0.698	0.813	0.858	0.885	0.922	0.936	0.951	0.982	0.988
D	0.055	0.244	0.482	0.661	0.741	0.787	0.854	0.880	0.908	0.967	0.977
	${}^bM_{ij}$ (7;8,t)										
NR2	0.095	0.095	0.209	0.387	0.469	0.511	0.603	0.673	0.754	0.911	0.939
D	0.000	0.126	0.422	0.645	0.731	0.780	0.849	0.875	0.905	0.966	0.977
	${}^bM_{ij}$ (4;5,t)										
NR2	0.125	0.250	0.383	0.447	0.463	0.594	0.703	0.747	0.772	0.815	0.840
D	0.000	0.063	0.183	0.325	0.390	0.541	0.681	0.738	0.769	0.821	0.848

Note: ${}^bM_i\ (v,s;t)$ ${}^bM_i$ ($v,s;\ t$) stands for the semi-Markov maintainabilities with initial backward starting from rating class *i*.

Table 6 Rating Dependent Cost of Capital

	AAA	AA	A	BBB	BB	B	CCC
(1+r) = 1.03	1.030	1.030	1.031	1.033	1.194	1.198	1.224

Note: r stands for the one period risk free interest rate.