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Forecasting VIX with Stock and Oil Prices

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Abstract

Using daily observations from 2004 to 2020, we find that separately, both stocks and oil price variables improve the prediction of the VIX, but not together. In particular, the oil price seems to be more informative. We study the sensitivity of our results with respect to different estimations setups; specifically we change the discounting factor in the EWLS (exponentially weighted least squares) estimation that seems to be relevant, but changing the size of the estimation window does not lead to unambiguous results. Finally, the numerical results show that the provided VIX forecasting models can help the investors to evaluate the volatility-related exchange traded products.

1. Introduction

The trading volumes of international futures and options are increasing annually. According to the Futures Industry Association (FIA), the trading volume of futures and options in 2021 is as high as 62,585 million, with an annual growth rate of 33.7%, of which the trading volume of options is approximately 33,309 million, with an annual growth rate of 56.6%. S&P 500 Index (SPX) options are more actively traded, with an average daily volume of approximately 1.37 million contracts. The VIX corresponds to an aggregate measure of implied volatility calculated by the Chicago Board Options Exchange (CBOE), based on SPX options; thus representing the expected volatility on the S&P 500 Index for the upcoming month. VIX is the main indicator for evaluating options and an important indicator to measure stock market risk (i.e., fear index). In addition, to meet investment and hedging needs, the CBOE issued VIX futures and options. Therefore, regardless of investment and hedging on stocks or derivatives, VIX estimation and forecasting are important topics. Unlike most previous studies, this study simultaneously uses stock and crude oil prices to predict the VIX. The stock price is represented by SPX, and the crude oil price is represented by West Texas Intermediate (WTI) crude oil.

Volatility is a measure of the standard deviation of the underlying asset returns, which usually changes over time (Tsay, 2010). There are two types of volatilities. The first is the historical volatility calculated from past return observations of underlying assets, and the second is the implied volatility derived from traded options prices given an options pricing formula, such as the Black-Scholes model (Hull, 2018). Since implied volatility is used to measure the market's volatility expectations for the future return of the underlying asset, it is called ex-post volatility (Harvey and Whaley, 1992). The Black-Scholes formula is derived under the assumption of constant volatility. However, implied volatility often changes with different

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moneyness levels. This phenomenon is called volatility smile or volatility skew. That is, implied volatility derived from low-strike price options is usually higher than that derived from high-strike price options (Hull, 2018). Notably, examining S&P 100 options from January 1984 to April 2006, Doran et al. (2007) evidenced that the volatility curve will be more skewed when the stock market crashes or spikes.

Many studies predict VIX or stock volatility. First, many studies have shown that the VIX has mean-reverting characteristics; for instance, the square root (SQR) model in Grünbichler and Longstaff (1996) and the log-normal Ornstein-Uhlenbeck (LOU) model in Dai and Singleton (2000). Second, Barndorff-Nielsen and Shephard (2000, 2001) introduced a model (called BN-S) to describe the joint behavior of stock returns and their volatility. Third, the HAR (heterogeneous autoregressive regression) model was frequently used to describe the realized volatility process (Corsi, 2009; Fernandes et al., 2014; Clements and Preve, 2021). Fourth, ARMA and GARCH family models have also been applied to capture VIX characteristics (Konstantinidi et al., 2008; Hao and Zhang, 2013; Wang et al., 2018). Although the prediction of VIX or volatility has been fully discussed in the literature, most of these models only use previous volatility as an explanatory variable. However, unlike the previous literature, we additionally provide explanatory variables (stock and oil prices) to predict the VIX.

Previous studies examine the relationship between stock returns and their volatilities. Fleming et al. (1995) find a large negative contemporaneous correlation between VXO changes and S&P 100 index returns, suggesting an inverse relationship between expected volatility and stock market prices. Similarly, Giot (2005) evidenced a strong negative relationship between contemporaneous changes in implied volatility indices and the underlying stock indices for both the S&P 100 and NASDAQ 100. Fassas and Siriopoulos (2021) confirmed the negative relationship between volatility indices and their underlying asset returns using 47 volatility indices as the sample. They also find a significant contemporaneous and mostly positive relationship between implied volatility changes and underlying returns. However, unlike most studies, Bollerslev and Zhou (2006) find that the correlation between SPX returns and implied volatility is unambiguously positive.

The forecast of the VIX or volatility contributes to the pricing reference of VIX derivatives (Demetrius et al., 1999; Mencía and Sentana, 2013; Griffin and Shams, 2018). Next, adding VIX derivatives to financial asset allocation can efficiently improve diversification (Szado, 2009; Kourtis et al., 2016). Additionally, VIX forecasts can be applied for risk management purposes, such as calculating value-at-risk (Kambouroudis et al., 2016). Moreover, volatility can be viewed as a relatively novel class of assets (Jabłecki et al., 2015; Latoszek and Ślepaczuk, 2020). Furthermore, choosing a suitable volatility forecast model can improve the performance of VIX futures trading strategies (Szado, 2019; Bilyk et al., 2020). Finally, although the VIX index is not tradable, the forecast of the VIX can help investors to evaluate its related derivative instruments such as the exchange-traded note VXX (Bašta and Molnár, 2019) or other volatility-related exchange traded products (Bordonado et al., 2017).

This study forecasted the VIX based on stock and crude oil prices, represented by SPX and WTI, respectively. Although many studies have predicted the VIX, few have considered both stock and oil prices in the prediction. The main conclusions of this study are as follows: We considered 20 regression models to predict VIX, where the explanatory variables included the one-day lag of VIX, SPX, and WTI. In addition to the random walk and AR(1) models, the other 18 regression models based on different explanatory variables can be divided into four categories: SPX-related models, WTI-related models, SPX+WTI-related models, and integrated models. Notably, the integrated models were combined with the random-walk model and the other categories. These regression models are estimated using the exponentially weighted least squares (EWLS) method, in which newer observations are assigned more weight than older ones. We adopted 20-, 40-, 60-, 80-, and 100-day moving windows to compare the VIX forecast performance of various regression models using the MAE (mean absolute error) and RMSE (root mean squared) as the predictive performance criteria. Unlike most previous studies, this study simultaneously considered the SPX and WTI to predict the one-day-ahead VIX using daily observations from 2004 to 2020 as the sample. In addition to comparing the forecast performance of various models, this study examined whether the forecast performance in predicting the VIX can remain unchanged with respect to the discount rate of the EWLS and moving window size. Finally, in addition to evaluating the forecast errors of the RMSE and MAE for various regression models, this study further compares the out-of-the-sample average returns in the VIX investment.

The remainder of this paper is organized as follows. Section 2 provides a literature review related to this study. Section 3 introduces the VIX forecasting models used in this study. Section 4 presents the empirical analysis. Finally, Section 5 presents our conclusions.

2. Literature Review

This section provides a literature review related to this study. We first introduce the CBOE VIX. We then review the literature focusing on the pricing of VIX derivatives and their investments.

2.1. CBOE VIX

The VIX, officially known as the Chicago Board Options Exchange (CBOE) Volatility Index, is the S&P 500 index volatility with risk-neutral expectations over the next 30 calendar days, reported annually (Mencía and Sentana, 2013; Miljkovic and SenGupta, 2018). In 1993, the CBOE launched a volatility index called the VXO index. Assuming σ is the annual implied volatility of the S&P 100 stock price index based on the Black-Scholes formula, VXO equals 100×σ (CBOE White paper, 2021). Subsequently, in 2003, the CBOE launched a new volatility index called the VIX index, where the VIX is calculated by evaluating variance swaps (Hull, 2018). The new VIX calculation formula is model-free and is not based on the Black-Scholes model or any other option pricing model (Jiang and Tian, 2007). Thus, estimation errors due to incorrect model specifications can be avoided.

CBOE formally calculates the VIX in real-time and updates the index every 15 seconds using SPX options with more than 23 and less than 37 days to expiration, where the SPX options include the at- and out-of-the-money vanilla call and put options (Griffin and Shams, 2018; Miljkovic and SenGupta, 2018). Referring to the CBOE VIX White Paper (2021), the VIX calculation formula is as follows:

$$\text{VIX} = 100 \times \sqrt{\frac{2}{T}\sum_{i}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2}, \quad (1)$$

where T is the time to expiration of options; F is the forward index level derived from index option prices; K_0 is the first strike price below F; $Q(K_i)$ is the out-of-the-money options price with strike price K_i, which equals call (put) price if $K_i > F$ $(K_i < F)$; $\Delta K_i = (K_{i+1} - K_{i-1})/2$; and r is the risk-free interest rate. The CBOE VIX is defined by the SPX options volatility with a maturity of 30 days. Accordingly, the next step in determining the VIX is to calculate the variances σ_1^2 and σ_2^2 for near-term and next-term options, respectively. Finally, σ^2 is calculated from the 30-day weighted average of σ_1^2 and σ_2^2, and thus, $\text{VIX} = 100 \times \sigma$.

2.2 Volatility or VIX Forecast

The VIX is an indicator of the fear index, which reflects the rise and fall of the stock market. Accordingly, accurate estimation of the VIX is very important, whether investing in the stock market or the pricing of derivative financial products of the VIX. Many studies predict the VIX or stock volatility, which can roughly be divided into one-, two-, and three-factor models.

First, two mean-reverting volatility models are widely used in the literature. In other words, the square root process (SQR) was proposed by Grünbichler and Longstaff (1996), whereas the log-normal Ornstein-Uhlenbeck (LOU) process was proposed by Dai and Singleton (2000). The SQR model specifies that the instantaneous movement of stock volatility is proportional to its square root, whereas the LOU model specifies that the instantaneous movement of stock volatility follows a log-normal distribution. Subsequently, the two mean-reverting models were extended further. Extending the SQR model, the concatenated SQR (CSQR) model proposed by Bates (2012) allows the VIX to revert toward a central tendency. LOUJ (LOU with a jump) model proposed by Mencía and Sentana (2013) adds a jump process with an exponential distribution to the LOU model. In addition, Mencía and Sentana (2013) introduced the LOU model with stochastic volatility (LOUSV) and the LOU model with central tendency and volatility (CTOUV). Mencía and Sentana (2013) found that the CSQR model could yield much smaller estimation errors than the SQR and LOU models, both in- and out-of-sample RMSEs. Additionally, adding a jump process to the LOU and CTOU models does not substantially improve their estimation performance.

Barndorff-Nielsen and Shephard (2000, 2001) introduced a model (called BN-S) to describe the joint behavior of stock returns and their volatility. The BN-S model captures the characteristics of financial time series with heavy-tailed distributions of log returns, aggregational Gaussianity, and quasi-long-range dependence (Mencía and Sentana, 2013). The BN-S model simultaneously specifies the probability distribution processes of stock returns and their variance. Given the current variance, the increments of variance and return are bivariate normally distributed with correlation coefficient ρ. However, the expected return (variance) increment is positively (negatively) proportional to the current variance. Subsequently, to detect the crude oil price jump size, Roberts and SenGupta (2020) added a random factor with a normal distribution to the BN-S model. That is, BN-S is a two-factor model, whereas the model (called R-SG model) proposed by Roberts and SenGupta (2020) is a three-factor model. In particular, a deterministic component in the R-SG model was extracted via machine and deep learning algorithms to improve the BN-S model. Additionally, Salmon and SenGupta (2021) introduced and analyzed the fractional Barndor-Nielsen and Shephard stochastic volatility models to evaluate arbitrage-free prices for variance and volatility swaps. They found that their proposed model outperformed the classical BN-S model. Moreover, SenGupta et al. (2021) proposed a model with machine learning algorithms to refine the BN-S model, which can efficiently improve the forecast ability in the crude oil price dynamics.

Corsi (2009) proposed a heterogeneous autoregressive regression (HAR) model, in which realized volatility is parameterized as a linear function of lagged realized volatilities over different horizons (day, week, and month). Fernandes et al. (2014) used the HAR model to predict the VIX and capture the long-range dependence of financial data. They found that the HAR model had a stronger VIX predictive ability than the random walk model and other models. Additionally, they find that the VIX is negatively related to SPX returns and positively related to the trading volume of the SPX. Clements and Preve (2021) believed that using raw realized variance and ordinary least squares (OLS) to estimate the HAR model is far from ideal. Accordingly, they investigated how the predictive accuracy of the HAR model depends on the choice of estimator, transformation, or combination scheme made by the market practitioner.

In addition to the above mean-reverting, BN-S family, and HAR models, numerous studies have also predicted the VIX, as summarized below. To forecast implied volatility, Konstantinidi et al. (2008) find that the ARIMA(1,1,1) and ARFIMA(1,d,1) models examining the American market are slightly better than the random walk model, and the vector autoregressive (VAR) and principal component analysis (PCA) models examining the European market significantly outperform the random walk model. Kambouroudis et al. (2016) compared the predictive ability of the GARCH family, implied volatility (IV), and realized volatility (RV) models in forecasting stock index return volatility. Using the RMSE and MAE criteria, they find that both the IV and RV forecasts contain significant information regarding future volatility. Additionally, previously implied volatility has a predictive ability to forecast future implied volatility. In particular, when the IV model accounts for the contemporaneous asymmetric effect, its forecast strictly outperforms the random walk model. Hao and Zhang (2013) found that the GARCH-implied VIX is significantly and consistently lower than the CBOE VIX when only returns are used for the estimation. Using high-frequency data, Qiao et al. (2020) found that the prediction performance of the DJI-GARCH (dynamic jump intensity GARCH) model is generally better than that of GARCH-type models. Wang et al. (2018) found that realized oil return volatility is predictive of realized stock return volatility. They find that simple linear regression is sufficient to capture the predictive relationships between crude oil and stock volatility. Paye (2012) performed predictive regressions for aggregate stock market volatility using macroeconomic variables and found that several variables related to macroeconomic uncertainty, time-varying expected stock returns, and credit conditions can Granger-cause volatility.

Although the prediction of VIX or volatility has been fully discussed in the literature, most of these models only use the previous volatility as an explanatory variable, similar to the prediction of VIX with an autoregressive model. Therefore, unlike previous literature, this study additionally provides explanatory variables (stock and oil prices) to predict the VIX.

2.3 VIX Derivatives Pricing and Investment

The forecast of the VIX or volatility contributes to the pricing reference for VIX futures, volatility, and variance swaps. In addition, because the VIX is a fear indicator, the VIX forecast can also provide a reference for stock market investment and risk management.

Demetrius et al. (1999) showed that variance swaps can theoretically be replicated by hedging portfolios of standard options with appropriate option strike prices. They derive analytic formulas for the theoretical fair value in the presence of realistic volatility skew. Szado (2009) finds that VIX calls can provide more efficient diversification than SPX calls during the 2008 financial crisis. Mencía and Sentana (2013) conducted an extensive empirical analysis of the VIX derivative valuation models before, during, and after the 2008–2009 financial crisis. They found that a process for the log of the observed VIX, combining central tendency and stochastic volatility, reliably priced VIX derivatives. Because the VIX calculation is primarily based on the volume of out-of-the-money options, the peak in volume for out-of-the-money SPX options typically occurs during the daily options settlement time. Griffin and Shams (2018) argue that the VIX settlement is susceptible to manipulation, leading to large transient deviations in prices. This indicates that the VIX settlement system is crucial.

Kourtis et al. (2016) constructed investment portfolios on international stock indices in which the investment weights were proportional to the reciprocals of stock index variances from 10 countries. They find that the Sharpe ratio (mean to standard deviation) of the portfolio using the VIX as a volatility proxy is higher than that using the historical volatility model. Kambouroudis et al. (2016) compare the predictive performance of implied volatility (IV), realized volatility (RV), and GARCH volatility in forecasting value-at-risk (VaR). Similarly, examining 13 stock indices, Kourtis et al. (2016) compare the prediction performance of implied volatility, realized volatility, and GARCH volatility in forecasting realized stock return volatility. Realized volatility is represented by the root of the sum of the square intraday returns. Cheng and Fung (2012) examined the information content of model-free implied volatility estimates with respect to the options and futures markets in Hong Kong.

Jabłecki et al. (2015) view volatility as a relatively novel asset class and review forecasting volatility methods for pricing major classes of volatility derivatives. They point out that volatility derivatives are an important group of financial instruments that can be considered in investment strategies and portfolio optimization. Latoszek and Ślepaczuk (2020) believe that volatility (treated as a new asset class) may improve portfolio performance because of its negative correlation with most asset types. They compared the performance of portfolios with and without the use of VIX derivatives under the mean-variance and naïve diversification approaches, respectively. Bilyk et al. (2020) compared the performance of VIX futures trading strategies built using different GARCH model volatility forecasting techniques. Long and short signals for VIX futures are produced by comparing one-day-ahead volatility forecasts with the current volatility. Using daily data, they found that strategies based on the fractional GARCH-threshold GARCH (fGARCH-TGARCH) and GJR-GARCH specifications outperformed those based on GARCH and exponential GARCH (EGARCH). Szado (2019) made two main VIX futures trading strategy indexes, including long or short positions of one- and three-month futures, until the market closes on the day prior to the VIX futures morning expiration.

3. Forecasting VIX Models

3.1. Regression Model and Parameter Estimation

This study aims to accurately predict the VIX index with multiple linear regression models and compare their forecast performances using the data of daily prices of the VIX, SPX, and WTI crude oil prices (simply WTI thereafter) as the sample. We employed ordinary least squares (OLS) and exponentially weighted least squares (EWLS) methods to estimate the parameters of the regression models. Let y_t and $\mathbf{x}_t \equiv (x_{1t}, \cdots, x_{kt})'$ respectively represent the dependent and independent variables vector in period *t*. The corresponding local multiple linear regression model for period *t* is expressed as

$$y_t = \boldsymbol{\beta}_t' \mathbf{x}_t + \varepsilon_t = \beta_{1t} x_{1,t} + \cdots + \beta_{kt} x_{k,t} + \varepsilon_t, \tag{2}$$

where $\boldsymbol{\beta}_t \equiv (\beta_{1t}, \cdots, \beta_{kt})'$ is time-varying parameters to a local regression, and the error term ε_t is normally distributed.

Wang et al. (2020) used the TWLS (time-dependent weighted least squares) method to estimate the regression model, in which newer observations were assigned more weight than older ones. Using the sample period from January 1927 to December 2017, they found that models using TWLS estimation have stronger predictive power in forecasting stock returns than those using OLS estimation, especially during periods of structural breaks. Notably, the EWLS method is a type of TWLS method. Referring to Hastie and Loader (1993), Grillenzoni (1999, 2008), Timmermann (2006), and Wang et al. (2021), this study adopts the EWLS method to estimate time-varying betas ($\boldsymbol{\beta}_t$).

To reflect the newer data to estimate $\boldsymbol{\beta}_t$ for each period, according to Bilyk et al. (2020), this study estimated the regression model with the observations of the most recent T days, where T is the moving window size. Additionally, to reflect the time-varying betas $\boldsymbol{\beta}_t$, rolling-based forecasting approaches (including expanding window and moving window approaches) were used to select the observations of each local regression. To allow each local regression to have equal observations and fairly analyze the VIX prediction performance of each regression model in each period, this study adopts a moving window approach.[2] Based on the EWLS method

[2] Expanding window includes all the previous observations until now, while the moving window includes

and the moving window approach, Equation (2) is written as the following local regression:

$$y_i = \boldsymbol{\beta}_t'\mathbf{x}_i + \varepsilon_i,\ \ \varepsilon_i \sim N(0, \sigma^2/\lambda^{t-i}),\ \ i = t, t-1, \ldots, t-T+1, \tag{3}$$

where λ is the discount rate, $0 < \lambda \leq 1$ and $1/\lambda^{t-i}$ represents the exponential weight of the *i*-th observation. The weight increased as i increased. This implies that the newer observation value has a larger weight. Thus, this study estimated $\boldsymbol{\beta}_t$ with the latest T daily data at t, t-1, …, t-T+1. Note that this study used moving windows of T = 20, 40, 60, 80, and 100 days to estimate the beta vector $\boldsymbol{\beta}_t$ in each local regression model. The weight is equal to 1 for the observation at time t and λ^{T-1} for the observation at time t-T+1. In particular, when λ = 1, Equation (3) conforms to the assumptions of the OLS method, indicating that all T observations are equally weighted.

In this study, the error term is assumed to follow a normal distribution with zero mean and variance of σ^2/λ^{t-i}, $\varepsilon_i \sim N(0, \sigma^2/\lambda^{t-i})$. This setting can make the estimators of $\boldsymbol{\beta}_t$ obtained by EWLS and GLS (generalized least squares) methods equal. In other words, the estimator of $\boldsymbol{\beta}_t$ is calculated to minimize the sum of the weighted square errors:

$$\widehat{\boldsymbol{\beta}}_t^{\text{EWLS}} = \arg\min_{\boldsymbol{\beta}_t} Q_t(\boldsymbol{\beta}_t) = \sum_{i=t-T+1}^{t} \lambda^{t-i} (y_i - \boldsymbol{\beta}_t'\mathbf{x}_i)^2 \tag{4}$$

Soling Equation (4) yields the EWLS estimator of $\boldsymbol{\beta}_t$,

$$\widehat{\boldsymbol{\beta}}_t^{\text{EWLS}} = (\textstyle\sum_{i=t-T+1}^{t} \lambda^{t-i}\mathbf{x}_i\mathbf{x}_i')^{-1}(\sum_{i=t-T+1}^{t} \lambda^{t-i}\mathbf{x}_i y_i). \tag{5}$$

Substituting $\lambda = 1$ into Equation (5) yields the OLS estimator of $\boldsymbol{\beta}_t$,

$$\widehat{\boldsymbol{\beta}}_t^{\text{OLS}} = (\textstyle\sum_{i=t-T+1}^{t} \mathbf{x}_i\mathbf{x}_i')^{-1}(\sum_{i=t-T+1}^{t} \mathbf{x}_i y_i). \tag{6}$$

The forecasting procedure in this study was divided into two steps. Step 1: Obtain the estimator of $\boldsymbol{\beta}_t$. Subsequently, Step 2 uses the estimated model at time t to predict the predicted variable y_{t+1} of the next period. First, this study intends to use the AR(1) model to predict the explanatory variables of the next period to capture short-term trends. Let x_t represent a single explanatory variable. Then, the AR(1) model is as follows:

$$x_i = \phi_{0t} + \phi_{1t}x_{i-1} + e_t,\qquad e_t \sim N(0, \sigma_e^2/\lambda^{t-i}),\qquad i = t, t-1, \ldots, t-T+1 \tag{7}$$

Similar to Equations (5) and (6), the EWLS and OLS estimators of $\boldsymbol{\phi}_t = (\phi_{0t}, \phi_{1t})'$ are expressed as follows.

only observations from the newer T days (moving window size) at each estimation (Bilyk et al., 2020).

$$\hat{\boldsymbol{\Phi}}_t^{\text{EWLS}} = \left(\sum_{i=t-T+1}^{t} \lambda^{t-i}\mathbf{z}_{i-1}\mathbf{z}_{i-1}'\right)^{-1} \left(\sum_{i=t-T+1}^{t} \lambda^{t-i}\mathbf{z}_{i-1}x_i\right) \tag{8}$$

and

$$\hat{\boldsymbol{\Phi}}_t^{\text{OLS}} = (\textstyle\sum_{i=t-T+1}^{t} \mathbf{z}_{i-1}\mathbf{z}_{i-1}')^{-1}(\sum_{i=t-T+1}^{t} \mathbf{z}_{i-1}x_i), \tag{9}$$

where the explanatory variable vector $\mathbf{z}_t = (1, x_t)'$. Thus, the EWLS or OLS predicted values of x_{t+1} is of the form:

$$\hat{x}_{t+1} = \hat{\phi}_{0t} + \hat{\phi}_{1t}x_t. \tag{10}$$

Hence, we can obtain the EWLS or OLS predicted values of y_{t+1}:

$$\hat{y}_{t+1} = \hat{\boldsymbol{\beta}}_t'\hat{\mathbf{x}}_{t+1}. \tag{11}$$

In this study, we use the VIX change (ΔV_{t+1}) as the explained variable, and the explanatory variables include the current VIX (V_t), the AR(1) predicted values of SPX return ($\hat{R}_{t+1}^S$), SPX change ($\widehat{\Delta S}_{t+1}$), WTI return ($\hat{R}_{t+1}^O$) and WTI changes ($\widehat{\Delta O}_{t+1}$). Additionally, based on our empirical data, this study found that the explained variable (ΔV_t) and the stock and oil price information variables (R_t^S, ΔS_t, R_t^O and ΔO_t) are the contemporaneous relationship. Actually, we found that the explanatory variables of $\hat{R}_{t+1}^S$, $\widehat{\Delta S}_{t+1}$, $\hat{R}_{t+1}^O$ and $\widehat{\Delta O}_{t+1}$ have greater forecast ability for ΔV_{t+1} than those of R_t^S, ΔS_t, R_t^O and ΔO_t. Finally, we intuitively judge that the EWLS method should have a stronger predictive performance than OLS because the EWLS method assigns greater weight to new observations and responds to the time-varying beta property.

3.2 Random Walk and AR(1) Models

The random walk model applies to the future trend of asset prices, which is random and unpredictable; therefore, the asset price in the current period becomes the best-predicted value of the asset price in the next period. Additionally, the AR(1) model applies to capture the short-term trend of asset prices. The random walk and AR(1) models for VIX forecasting are respectively expressed as follows.

$$\text{M0a: } \widehat{\Delta V}_{t+1} = 0$$

$$\text{M0b: } \widehat{\Delta V}_{t+1} = a_0' + b_0 V_t$$

Notably, past empirical studies often adopt the random walk model as a benchmark to evaluate whether other models have stronger forecasting asset price ability (Konstantinidi et al., 2008; Fernandes et al., 2014; Kambouroudis et al., 2016). Baba and Sakurai (2011) assumed that the VIX index follows the AR(1) process with Markov regime-switching parameters.

In this study, M0a and M0b are used as benchmark models for predicting the VIX to test whether the regression models using stock or oil price information as their explanatory variables can improve their ability to predict the VIX. Because we are making one-day-ahead forecasts, these explained and explanatory variables could likely have a short-term time series trend of the AR(1) process. Additionally, several mean-reverting volatility models have been widely used, such as the SQR proposed by Grünbichler and Longstaff (1996), LOU proposed by Dai and Singleton (2000), and CSQR proposed by Bates (2012). As in M0a and M0b models, these mean-reverting models have no explanatory variables other than the VIX or volatility itself. This implies that the forecasting ability of the random walk and AR(1) models should not be negligible.

3.3 Predicting the VIX Based on Stock and/or Oil Prices

Referring to Konstantinidi et al. (2008) and Fernandes et al. (2014), we adopted the SPX price to predict the VIX. In addition, we consider oil price information to predict VIX. Let V_t, S_t, R_t^S, O_t and R_t^O denote the VIX index, stock price, stock return, oil price and oil return at period *t*, respectively. At the significance level of 0.001, Table 2 in the next section confirms that the stock and oil prices and their logarithms belong to I(1), integrated with degree one (Greene, 2002). This implies that their first-order differences belong to I(0) and hence, are stationary. Accordingly, to avoid the spurious regression problem, their first-order differences were used as explanatory variables in this study. Because the first-order differences of logarithms are equal to their return rates, the stock and oil return rates are taken as the explanatory variables. In addition, the short-term movement of the VIX may be related to its one-day lag of VIX size, such as the mean-reverting behavior of the VIX proposed in previous literature (Grünbichler and Longstaff, 1996; Dai and Singleton, 2000; Bates, 2012). Consequently, the considered explanatory variables in forecasting VIX include the five stationary variables of V_t, ΔS_t, R_t^S, ΔO_t and R_t^O. In summary, this subsection considers 12 multiple linear regression models for forecasting VIX as follows:

M1a: $\widehat{\Delta V}_{t+1} = a_1 + a_2 \hat{R}_{t+1}^S$
M1b: $\widehat{\Delta V}_{t+1} = a_1' + a_2' \hat{R}_{t+1}^S + b_1 V_t$
M2a: $\widehat{\Delta V}_{t+1} = a_3 + a_4 \widehat{\Delta S}_{t+1}$
M2b: $\widehat{\Delta V}_{t+1} = a_3' + a_4' \widehat{\Delta S}_{t+1} + b_2 V_t$
M3a: $\widehat{\Delta V}_{t+1} = a_5 + a_6 \hat{R}_{t+1}^O$
M3b: $\widehat{\Delta V}_{t+1} = a_5' + a_6' \hat{R}_{t+1}^O + b_3 V_t$
M4a: $\widehat{\Delta V}_{t+1} = a_7 + a_8 \widehat{\Delta O}_{t+1}$
M4b: $\widehat{\Delta V}_{t+1} = a_7' + a_8' \widehat{\Delta O}_{t+1} + b_4 V_t$
M5a: $\widehat{\Delta V}_{t+1} = a_9 + a_{10} \hat{R}_{t+1}^S + a_{11} \hat{R}_{t+1}^O$
M5b: $\widehat{\Delta V}_{t+1} = a_9' + a_{10}' \hat{R}_{t+1}^S + a_{11}' \hat{R}_{t+1}^O + b_5 V_t$
M6a: $\widehat{\Delta V}_{t+1} = a_{12} + a_{13} \widehat{\Delta S}_{t+1} + a_{14} \widehat{\Delta O}_{t+1}$
M6b: $\widehat{\Delta V}_{t+1} = a_{12}' + a_{13}' \widehat{\Delta S}_{t+1} + a_{14}' \widehat{\Delta O}_{t+1} + b_6 V_t$

Compared with M1a through M6a, the explanatory variables of M1b through M6b add a one-day lag of VIX (V_t) to the explanatory variables to capture the short-term memory of VIX. Several previous studies have adopted V_t to predict V_{t+1}. For example, Kambouroudis et al. (2016) used the ARMA(1,1) and ARIMA (1,1,1) models to predict the VIX index. Additionally, the SPX-related models include M1a, M1b, M2a, and M2b, whereas the WTI-related models include M3a, M3b, M4a, and M4b. Moreover, the SPX+WTI-related models include M5a, M5b, M6a, and M6b. Furthermore, M1a, M3a, and M5a take stock and oil returns as explanatory variables, while M2a, M4a, and M6a take the movements of stock and oil prices as explanatory variables.

3.4 Integrated Models

Given the current time t, we want to forecast the dependent variable in time t + h, y_{t+h}. Assume that we obtain N forecast values of y_{t+h}, $\hat{\mathbf{y}}_{t+h,t}$, corresponding to *N* different estimating methods. That is,

$$y_{t+h} = \hat{y}_{t+h,i} + e_{t+h,i}\,,\ i = 1, 2, \dots, N, \tag{12}$$

where $e_{t+h,i}$ is the residual for the i-th estimation method. Granger and Ramanathan (1984) proposed the following integrated regression model.

$$y_{t+h} = \mathbf{w}'\hat{\boldsymbol{y}}_{t+h} + \varepsilon_{t+h} = w_1\hat{y}_{t+h,1} + \cdots + w_N\hat{y}_{t+h,N} + \varepsilon_{t+h}, \tag{13}$$

where $\mathbf{w}'\mathbf{1} = w_1 + \cdots + w_N = 1$ and the weighted vector $\mathbf{w}$ is estimated by the OLS method.

Additionally, Stock and Watson (2001) and Timmermann (2006) proposed a broader set of combination weights that also ignore correlations between forecast errors but base the combination weights on the models' relative mean square error (MSE) performance raised to various powers. Let $MSE_{t+h,i}$ be the MSE of the ith forecasting model at time t. Subsequently, using a moving window size *h* at time t, the integrated predicted value of y_{t+h} is computed as follows:

$$\hat{y}_{t+h} = \widehat{w}_1\hat{y}_{t+h,1} + \cdots + \widehat{w}_N\hat{y}_{t+h,N},\ \widehat{w}_i = \frac{1/\mathrm{MSE}_{t+h,i}}{\sum_{j=1}^{N}(1/\mathrm{MSE}_{t+h,j})}. \tag{14}$$

Moreover, Miljkovic and SenGupta (2018) proposed a *K*-component mixture of regressions model to analyze S&P 500 market fluctuations. Consider *K* multiple regression models with the same dependent and independent variables, as follows.

$$\boldsymbol{y} = \mathbf{X}\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i \text{ with probability } \pi_i,\ i = 1, \dots, K, \tag{15}$$

where $0 \le \pi_i \le 1$ and $\sum_{i=1}^{K}\pi_i = 1$. Miljkovic and SenGupta (2018) applied the expectation maximization (EM) algorithm to estimate the parameters of $\{\pi_i\}_{i=1}^{K}$ and $\{\boldsymbol{\beta}_i\}_{i=1}^{K}$. The EM approach is similar to the concept of MLE and can be regarded as a weighted MLE method for *K* regression models. Thus, the *K*-component mixture of the regression model can be regarded as an integrated model.

Compared with Granger and Ramanathan (1984) and Miljkovic and SenGupta

(2018), the model in Equation (14) is easier to handle and matches some machine learning senses. Therefore, the integrated model used in this study is based on Stock and Watson (2001) and Timmermann (2006). In other words, the weight of each regression model in the integrated model is proportional to the reciprocal of the forecast MSE.

Since the random walk model usually has good predictive performance for the VIX, we combined the random walk model with the aforementioned models M1a–M6b to design six different integrated models. Integrated models were constructed in two steps. The first step combines M1a through M6b, and the second step integrates the combined models with the random-walk model. Accordingly, the first step considers six combined models as follows:

$$\begin{cases} \hat{V}^{a}_{S,t+1} = c_1\hat{V}^{1a}_{t+1} + c_2\hat{V}^{2a}_{t+1} \\ \hat{V}^{b}_{S,t+1} = c'_1\hat{V}^{1b}_{t+1} + c'_2\hat{V}^{2b}_{t+1} \\ \hat{V}^{a}_{O,t+1} = c_3\hat{V}^{3a}_{t+1} + c_4\hat{V}^{4a}_{t+1} \\ \hat{V}^{b}_{O,t+1} = c'_3\hat{V}^{3b}_{t+1} + c'_4\hat{V}^{4b}_{t+1} \\ \hat{V}^{a}_{SO,t+1} = c_5\hat{V}^{1a}_{t+1} + c_6\hat{V}^{2a}_{t+1} + c_7\hat{V}^{3a}_{t+1} + c_8\hat{V}^{4a}_{t+1} \\ \hat{V}^{b}_{SO,t+1} = c'_5\hat{V}^{1b}_{t+1} + c'_6\hat{V}^{2b}_{t+1} + c'_7\hat{V}^{3b}_{t+1} + c'_8\hat{V}^{4b}_{t+1} \end{cases} \tag{16}$$

where $\hat{V}^{1a}_{t+1}$ through $\hat{V}^{4a}_{t+1}$ represent the predicted VIX values estimated by M1a through M4a, and $\hat{V}^{1b}_{t+1}$ through $\hat{V}^{4b}_{t+1}$ represent the predicted VIX values estimated by M1b through M4b. Additionally, the coefficients in Equation (16) denote the weight of the model, and the size of the weight is inversely proportional to the square of the prediction error. We consider $\hat{V}^{a}_{SO,t+1}$ as an example to calculate its coefficients (c_5, c_6, c_7 and c_8). Let e_5 through e_8 represent the predictive error (predicted value minus actual value) of M1a through M4a for the VIX in the previous period. Then,

$$c_i = \frac{e_i^{-2}}{e_5^{-2} + e_6^{-2} + e_7^{-2} + e_8^{-2}}, \quad i = 5, 6, 7, 8. \tag{17}$$

Subsequently, combing the models in Equation (17) with the random walk model, this study considers six integrated models, as follows.

$$\begin{aligned} &\text{M7a: } \hat{V}_{t+1} = \omega_{7a} \times \hat{V}^{RW}_{t+1} + (1-\omega_{7a}) \times \hat{V}^{a}_{S,t+1} \\ &\text{M7b: } \hat{V}_{t+1} = \omega_{7b} \times \hat{V}^{RW}_{t+1} + (1-\omega_{7b}) \times \hat{V}^{b}_{S,t+1} \\ &\text{M8a: } \hat{V}_{t+1} = \omega_{8a} \times \hat{V}^{RW}_{t+1} + (1-\omega_{8a}) \times \hat{V}^{a}_{O,t+1} \\ &\text{M8b: } \hat{V}_{t+1} = \omega_{8b} \times \hat{V}^{RW}_{t+1} + (1-\omega_{8b}) \times \hat{V}^{b}_{O,t+1} \\ &\text{M9a: } \hat{V}_{t+1} = \omega_{9a} \times \hat{V}^{RW}_{t+1} + (1-\omega_{9a}) \times \hat{V}^{a}_{SO,t+1} \\ &\text{M9b: } \hat{V}_{t+1} = \omega_{9b} \times \hat{V}^{RW}_{t+1} + (1-\omega_{9b}) \times \hat{V}^{b}_{SO,t+1} \end{aligned}$$

where $\hat{V}^{RW}_{t+1}$ represents the VIX value predicted by the random walk model and the coefficients ω_{7a}, ω_{7b}, ω_{8a}, ω_{8b}, ω_{9a} and ω_{9b} represent the weighted values of the random walk model in M7a, M7b, M8a, M8b, M9a, and M9b, respectively. As in Equation (17), their weight sizes are inversely proportional to the square of the forecast errors in the previous period.

3.5 Performance Evaluation

Based on Konstantinidi et al. (2008), Fernandes et al. (2014), Kambouroudis et al. (2016), Kourtis et al. (2016), Qiao et al. (2020), and Bilyk et al. (2020), this study adopted the mean absolute error (MAE), and root mean squared error (RMSE) as the predictive performance measures of various models for forecasting VIX. The calculation formulas for MAE and RMSE are expressed as follows:

$$\text{MAE} = \frac{1}{N}\sum_{t=T+1}^{T+N}\left|\hat{V}_t - V_t\right| \text{ and } \text{RMSE} = \sqrt{\frac{1}{N}\sum_{t=T+1}^{T+N}\left(\hat{V}_t - V_t\right)^2} \tag{18}$$

where *N* is the number of out-of-sample observations, V_t is the realized value of the VIX, and $\hat{V}_t$ is the predicted value from various predictive models.

Hansen et al. (2011) introduced the model confidence set (MCS) and applied it to the selection of models. An MCS is a set of models constructed such that it contains the best model with a given level of confidence. They apply the MCS procedure to the inflation forecasting problem of Stock and Watson (1999). They found that the MCS procedure provides a powerful tool for evaluating competing inflation forecasts using the RMSE as the forecast performance criterion. Although the MCS method can effectively evaluate various regression models, it is not mainly applied to the moving window approach and hence, is not applied in this study.

In addition to evaluating the forecast errors of RMSE and MAE for various regression models, this study further compares their out-of-the-sample average returns in VIX investment. The investment strategy in Bilyk et al., (2020), long and short signals for VIX futures are produced by comparing one-day ahead volatility forecasts with current volatility. Referring to Bilyk et al., (2020), the investment strategy in this study is that we buy (sell) if the one-day ahead VIX forecast is larger (smaller) than the current VIX price. That is, the net position on VIX is equal to 1, 0, and -1 if the next-day VIX price is expected to rise, remain unchanged and fall, respectively.

4. Empirical Analysis

4.1 Data Description

This study aims to predict the VIX index using the daily price data of the VIX, SPX, and WTI crude oil prices (simply WTI thereafter) as the sample. Since the CBOE launched the new VIX on September 22, 2003, the sample period of this study was from January 2, 2004, to December 31, 2020. VIX data were obtained from the CBOE website, and SPX and WTI oil prices were extracted from the Federal Reserve Economic Database of the Bank of St. Louis (https://fred.stlouisfed.org/). The original sample comprised 4,280 daily observations. Since the oil price on April 20, 2020, was negative, its logarithmic value and return could not be calculated.

Therefore, the observations on that day were deleted, and the final effective sample consisted of 4,279 daily data points. Currently, the crude oil markets with the largest trading volumes in the world include West Texas crude oil (WTI), North Sea Brent Crude Oil (Brent) in Europe, and Dubai crude oil in the Middle East (Dubai). As this study focuses on the US market, WTI oil was selected as the representative crude oil price. This study employed EXCEL and EViews software for data treatment and analysis and used MATLB to perform VIX prediction based on various regression models.

Table 1 reports the daily summary statistics of the VIX, SPX, and WTI. During the sample period, the VIX ranged from 9.14 to 82.69, with an average value of 18.86 and standard deviation of 9.22; the SPX ranged from 676.53 to 3756.07, with an average of 1772.76 and standard deviation of 690.35; and the WTI ranged from \$8.91 to \$145.31 per barrel, with an average of \$69.36 and standard deviation of \$23.14. Additionally, to determine stationarity, Table 2 lists the augmented Dickey-Fuller (ADF) unit root test results for the three variables. We found that V_t, ΔS_t, ΔO_t, $\Delta \ln S_t$ and $\Delta \ln O_t$ have not unit roots at a 0.001 significant level. The results reveal that VIX is a stationary time series, that is, $V_t \in I(0)$. Although SPX (S_t) and WTI (O_t) have unit roots, their first differences are stationary, that is, $S_t \in I(1)$ and $O_t \in I(1)$. Additionally, since $R_t^S = \Delta \ln S_t$ and $R_t^O = \Delta \ln O_t$, $R_t^S \in I(0)$ and $R_t^O \in I(0)$. Notably, the fact that the VIX is a stationary time series is consistent with Kambouroudis et al. (2016), who tested VIX data from February 2, 2001, to February 28, 2013. Moreover, to avoid the spurious regression problem, this study adopts the stationary explanatory variables (V_t, ΔS_t, ΔO_t, R_t^S, R_t^O) to forecast the next period change of VIX (ΔV_{t+1}).

Figure 1 depicts the trend charts for the VIX, SPX, and WTI. The VIX trend is roughly opposite to that of the SPX; therefore, the VIX is often regarded as a fear index in financial markets. During the sample period, the VIX was affected by the US subprime mortgage storm, which caused the VIX to reach its all-time high of 80.86 on November 20, 2008. At this time, the SPX fell to a relatively low level of 752.44, and the oil price fell to a relatively low level of \$48.86 per barrel on that day. Second, due to the recent impact of the COVID-19 epidemic, the VIX hit a record high of 82.69 on March 16, 2020, and the SPX on that day plummeted by nearly 325 points (closing price was 2386.13). Figure 1 shows that the VIX and SPX have an inverse relationship over the same period; that is, when VIX rises, the SPX will fall simultaneously. Additionally, the empirical analysis results of Wang et al. (2018) indicate that large fluctuations in crude oil always occur together with large fluctuations in stocks, even before large fluctuations in stocks. From the trend charts of the VIX and WTI, we can observe that oil prices seemed to react earlier than the VIX with respect to the large fluctuations in stocks from 2008 to 2009.

Table 3 displays the descriptive statistics of the variables in this study, including their average, standard deviation, and maximum, minimum, and correlation coefficients. The VIX-related variables include the original value of the VIX (V_t) and its daily change (ΔV_{t+1}); the SPX-related variables include the SPX daily return (R_t^S), predictive return ($\hat{R}_{t+1}^S$), daily change (ΔS_t) and predictive daily change ($\widehat{\Delta S}_{t+1}$); and the WTI-related variables include the WTI daily return (R_t^O), predictive daily return ($\hat{R}_{t+1}^O$), daily change (ΔO_t) and predictive daily change ($\widehat{\Delta O}_{t+1}$). Note that $\hat{R}_{t+1}^S$,

$\widehat{\Delta S}_{t+1}$, $\hat{R}^O_{t+1}$ and $\widehat{\Delta O}_{t+1}$ are the predictive values of $\hat{R}^S_t$, $\widehat{\Delta S}_t$, $\hat{R}^O_t$, and $\widehat{\Delta O}_t$, respectively, based on the AR(1) model for a moving window size of T = 60 days. Figure 2 shows the scatter diagram of these variables. Table 3 and Figure 2 reveal the following primary findings: First, VIX (V_t) is very close to its one-day lag value (V_{t-1}), indicating that V_t is likely a random-walk process. Second, ΔV_{t+1} is slightly negatively related to $\hat{R}^S_{t+1}$ and $\widehat{\Delta S}_{t+1}$ with correlation coefficients of -0.0669 and -0.0705, respectively. This implies that the SPX-related regression models of M2a and M2b are unlikely to have good predictive ability for VIX. Third, ΔV_{t+1} is slightly positively related to $\hat{R}^O_{t+1}$ and $\widehat{\Delta O}_{t+1}$ with correlation coefficients of 0.0265 and 0.0061, respectively. Fourth, ΔV_{t+1} is weakly negatively related to V_t, with a correlation coefficient of -0.1022. Finally, ΔV_{t+1} is negatively related to R^S_t and R^O_t, with correlation coefficients of -0.8246 and -0.2465, respectively, which is consistent with the results of Fleming et al. (1995) and Giot (2005). Finally, Figure 2 shows that ΔV_t is clearly negatively related to R^S_t and ΔS_t, indicating that the VIX is a fear index for the stock return market. However, Elder and Serletis (2010) noted that owing to the importance of crude oil in the US economy, an increase in crude oil volatility will lead to stronger macroeconomic uncertainty and greater stock volatility. Therefore, we expected oil prices to be informative in the VIX forecast.

This study uses the AR(1) model to predict the explanatory variables of the next period in order to capture there short-term trend. Table 3 displays that the correlation coefficients of ($\hat{R}^S_{t+1}$, R^S_t), ($\widehat{\Delta S}_{t+1}$, ΔS_t), ($\hat{R}^O_{t+1}$, R^O_t) and ($\widehat{\Delta O}_{t+1}$, ΔO_t) are equal to 0.0909, -0.0972, -0.0074 and 0.01329, respectively. Notably, these correlation coefficients are equivalent to their first-order autoregression coefficients. Under the null hypothesis that the first-order autoregression coefficient equals zero, the corresponding t-test statistics for these correlation coefficients equals 5.9681, 6.3855, 0.4839, 0.86518, respectively.[3] This indicating that the correlation coefficients of ($\hat{R}^S_{t+1}$, R^S_t) and ($\widehat{\Delta S}_{t+1}$, ΔS_t) are significantly different from zero, while the other two are not. Nevertheless, whatever their significances, this study found that the explanatory variables of $\hat{R}^S_{t+1}$, $\widehat{\Delta S}_{t+1}$, $\hat{R}^O_{t+1}$ and $\widehat{\Delta O}_{t+1}$ have greater forecast power for ΔV_{t+1} than the explanatory variables of R^S_t, ΔS_t, R^O_t and ΔO_t.

[3] The t-test statistics equals $t = r\sqrt{(n-2)/(1-r^2)}$, where *r* and *n* (= 4275 herein) represent the correlation coefficient and the number of observations, respectively.

Table 1 Daily Summary Statistics of VIX, SPX, and WTI

	VIX	*SPX*	*WTI*
Mean	18.86	1772.76	68.36
S.D	9.22	690.35	23.14
Max	82.69	3756.07	145.31
Min	9.14	676.53	8.91
Correlation matrix			
VIX	1.0000		
SPX	0.9730	1.0000	
WTI	0.9675	0.9963	1.0000

Notes: There are 4279 daily data over the sample period from January 2, 2004 to December 31, 2020.

Table 2 Augmented Dickey-Fuller Unit Root Test

	Level		*First Difference*	
	Constant	*Constant and Trend*	*Constant*	*Constant and Trend*
V_t	-4.9229***[0.0000]	-4.9239***[0.0003]		
S_t	1.3392 [0.9989]	-0.9707 [0.9462]	-13.4916***[0.0000]	-13.6438***[0.0000]
lnS_t	0.2036 [0.9729]	-0.7776 [0.7157]	-15.9632***[0.0000]	-16.0063***[0.0000]
O_t	-2.3123 [0.1681]	-2.5343 [0.3113]	-29.6415***[0.0000]	-29.6569***[0.0000]
lnO_t	-3.2864 [0.0156]	-3.5299 [0.0363]	-10.6885***[0.0000]	-10.7134***[0.0000]

Notes: There are 4279 daily data over the sample period from January 2, 2004 to December 31, 2020. V_t, S_t and O_t denote the prices of VIX, SPX, and WTI crude oil, respectively. The notation "***" denotes the 1‰ two-tailed test significance for the null hypothesis of the existing unit root. The lag length for the test is set to minimize AIC. These numbers are t-values with p-values in parentheses [].

Figure 1 The Trend Charts of VIX, SPX, and WTI

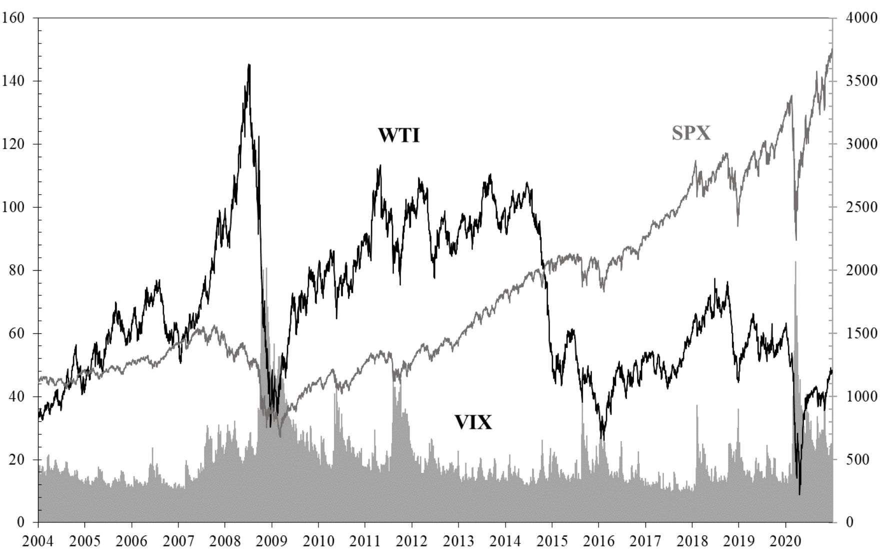
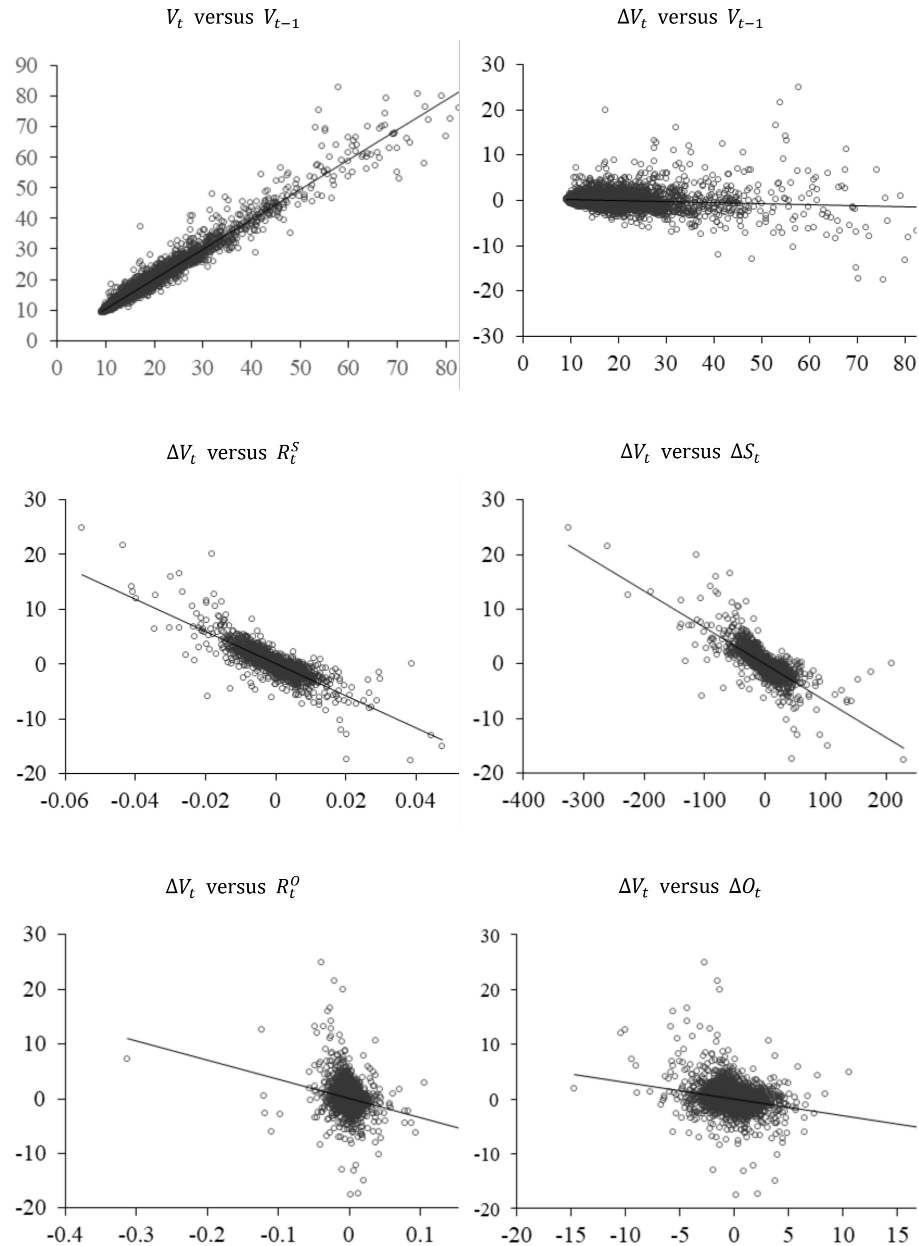


Table 3 Descriptive Statistics of Variables

	ΔV_{t+1}	V_t	R_t^S	$\hat{R}_{t+1}^S$	ΔS_t	$\widehat{\Delta S}_{t+1}$	R_t^O	$\hat{R}_{t+1}^O$	ΔO_t	$\widehat{\Delta O}_{t+1}$
Mean	0.0014	18.8957	0.0004	0.0002	0.6242	0.4354	0.0005	-0.0002	0.0033	-0.0041
S.D	1.8984	9.2773	0.0123	0.0032	22.0166	6.4418	0.0301	0.0094	1.5751	0.3584
Max	24.8600	82.6900	0.1158	0.0641	230.3800	138.5457	0.5309	0.0915	18.5600	4.3076
Min	-17.6400	9.1400	-0.1198	-0.0579	-324.8900	-127.4487	-0.5134	-0.3380	-14.7600	-8.8020
Correlation matrix										
ΔV_{t+1}	1.0000									
V_t	-0.1022	1.0000								
R_t^S	-0.8246	0.0397	1.0000							
$\hat{R}_{t+1}^S$	-0.0669	-0.2345	0.0909	1.0000						
ΔS_t	0.1443	-0.0037	-0.1471	-0.0748	1.0000					
$\widehat{\Delta S}_{t+1}$	-0.0705	-0.1896	0.0904	0.9341	-0.0972	1.0000				
R_t^O	-0.2465	-0.0095	0.2512	0.0618	-0.0475	0.0886	1.0000			
$\hat{R}_{t+1}^O$	0.0265	-0.1978	-0.0243	0.1426	0.0685	0.1514	-0.0074	1.0000		
ΔO_t	-0.0456	-0.0238	0.0528	0.0095	0.2383	-0.0044	-0.0417	-0.0089	1.0000	
$\widehat{\Delta O}_{t+1}$	0.0061	-0.2784	0.0080	0.2207	0.0346	0.1792	-0.0216	0.7731	0.0132	1.0000

Notes: There are totally 4279 daily data over the sample period from January 2, 2004 to December 31, 2020. V_t, S_t and O_t denote the prices of VIX, SPX and WTI crude oil, respectively. R_t^S and R_t^O represent the continuously daily returns of SPX and WTI crude oil, respectively. Note that $\hat{R}_{t+1}^S$, $\widehat{\Delta S}_{t+1}$, $\hat{R}_{t+1}^O$ and $\widehat{\Delta O}_{t+1}$ are the predictive values of $\hat{R}_t^S$, $\widehat{\Delta S}_t$, $\hat{R}_t^O$ and $\widehat{\Delta O}_t$ based on the AR(1) model to the moving window size of T = 60 days.

Figure 2 Scatter Diagram for VIX



4.2 Forecast Ability Comparison

This subsection empirically compares the forecast ability and investment performance of the VIX among the 20 predictive regression models in Section 3. Additionally, this study compares whether the moving window size (T) and discount rate (λ) can affect forecast ability and investment. Accordingly, the moving window size is separately set to T = 20, 40, 60, 80, and 100 days,[4] whereas the discount rate is separately set to λ = 0.30, 0.35, …, 0.95, and 1.00, respectively, for the EWLS approach. Note that the EWLS approach is the same as the OLS approach when λ = 1.00. Additionally, to fairly compare their forecasts among different moving window sizes, regardless of the moving window size, the out-of-the-sample observation period was adopted to evaluate the forecast performance range from May 28, 2004, to December 31, 2020. Thus, the out-of-the-sample has 4177 observations, which corresponds to a moving window size of 100 and a one-day-ahead forecast of $\widehat{\Delta S}_{t+1}$ and $\widehat{\Delta O}_{t+1}$ $(4279 - 100 - 2)$. In particular, this subsection only considers the moving window T = 60 as the representative to compare the predictive performance of various models. Subsequently, the next subsection focuses on comparing their forecast performance against different moving window sizes and discount rates. Tables 4 through 6 report the forecast MAE, RMSE, and mean-return for various predictive regression models with a moving window of T = 60.

Random Walk and AR(1)

We use the random walk or AR(1) model as a benchmark to compare the forecasting ability of various regression models. Table 4 shows that the RMSE of M0a is 1.9051, which is larger than 1.8244, the average of M0a to M9a. Additionally, the average RMSE of M0b is 3.2472, which is larger than 1.8903, the average of M0b–M9b. Next, Table 5 reports that the MAE of M0a is 1.0797, which is larger than 1.0005, the average of M0a to M9a. Additionally, the average MAE of M0b is 1.3230, which is larger than 0.9842, the average of M0b–M9b. The above results indicate that the forecast ability of the random walk or AR(1) model on the VIX is not desirable, and hence the stock and oil prices are informative in predicting the VIX.

The RMSE in the M0b (AR(1)) model is stably decreasing with the discount rate, ranging from 2.2442 at $\lambda = 1.0$ to 4.8290 at $\lambda = 0.3$. This fact indicates that given the AR(1) model to predict VIX, using the OLS method ($\lambda = 1$) to estimate the regression parameters is not inferior to using the EWLS method. A possible reason for this is as follows. Relative to OLS, EWLS gives a larger weight to newer observations to capture the short-term dynamic behavior of the VIX. However, the explained variable in M0b is a one-day lag of the VIX, which can be used to capture its short-term dynamic behavior. Accordingly, if we assign a larger weight to newer observations, this leads to overly ignoring information for older observations. On the other hand, when MAE is taken as the performance criterion, Table 5 displays that the MAE for the M0b model is decreasing from 1.1706 at $\lambda = 0.3$ to 1.1910 at

[4] The moving window is set to be T = 15 or 25 in Grillenzoni (1999), T = 21 in Bilyk et al. (2020), and T = 30, 60, …, 240 in Wang (2021), respectively.

$\lambda = 0.85$, then increasing to 1.2235 at $\lambda = 1.0$. This indicates that the MAE for the M0b model is a U-shaped function of the discount rate.

One-Day Lag of VIX

Konstantinidi et al. (2008) found that the AR(1) and ARIMA(1,1,1) models in predicting VIX in the American market are slightly better than the random walk model. Additionally, Kambouroudis et al. (2016) found that the predictive performance of the ARMA(1,1) model for the VIX was also higher than that of the random walk model. Accordingly, this study intends to compare the predictive performance of models in which the explanatory variables contain a one-day lag of VIX (M1b through M9b) and the others (M1a through M9a). When the regression models are estimated by the OLS method, the average RMSE (MAE) of M1b through M9b is equal to 1.8718 (1.0796), which is lower than 1.9187 (1.0888) of M1b through M9b. Additionally, when the regression models are estimated using the EWLS method, adding the one-day lag of VIX into these regression models can decrease their MAEs, except for the models of M5b at λ = (0.30, 0.40, 0.45) and M6b at λ = (0.30, 0.35, 0.40). In short, these facts indicate that using a one-day lag in VIX information helps predict the next day's VIX. This finding is consistent with the results of Kambouroudis et al. (2016).

Stock and Oil Prices

We now analyze the forecast performances of the SPX-related models (M1a, M1b, M2a, M2b), WTI-related models (M3a, M3b, M4a, M4b), and SPX+WTI-related models (M5a, M5b, M6a, M6b). First, M1b–M6b have smaller forecast errors in RMSE and MAE than those in M0b for all discount rates. Additionally, M1a through M6a have smaller forecast errors in RMSE and MAE than M0a for the discount rate $\lambda \leq 0.90$. This result reveals that stock and oil prices are informative in predicting the VIX. Second, the forecast abilities of the RMSE and MAE of M1a, M1b, M3a, and M3b were roughly equal to those of M2a, M2b, M4a, and M4b, respectively. This implies that the explanatory abilities of stock and oil returns in predicting the VIX are almost equal to those of stock and oil price changes. Third, WTI-related models (M3a, M3b, M4a, and M4b) generally have better forecasting ability than SPX-related models (M1a, M1b, M2a, and M2b), except M3a at λ = 1.0. That is, the average RMSE (MAE) for M3a, M3b, M4a, and M4b were 1.7222 (0.9187), 1.5435 (0.8052), 1.7022 (0.9169), and 1.5178 (0.8022), respectively, whereas the average RMSE (MAE) for M1a, M1b, M2a, and M2b were 1.8836 (1.0261), 1.8521 (1.0066), 1.8809 (1.0252), and 1.8453 (1.0048), respectively. Accordingly, relative to stock prices, oil prices have a better explanatory ability to forecast the VIX. This result is consistent with the empirical findings of Paye (2012) and Wang et al. (2018), who pointed out that oil price fluctuations have a high predictive ability for stock volatility. Fourth, the average RMSE (MAE) values were 1.9076 (1.0548), 1.8960 (1.0455), 1.9065 (1.0549), and 1.8905 (1.0444) for M5a, M5b, M6a, and M6b, respectively. This result reveals that the SPX+WTI-related models generally have larger forecast errors than the SPX-related and WTI-related models. In general, using too many explanatory variables in a regression model can decrease its forecasting ability. This is known as the overfitting problem. Accordingly, using the stock or oil price variables alone helps predict the VIX, but using both reduces the predictive power. This result is consistent with Konstantinidi et al. (2008), in which the predictive model with nine economic explanatory variables has a larger forecast error in the RMSE than the random walk model.

Integrated Model

This study considers six integrated models (M7a, M7b, M8a, M8b, M9a, and M9b), whose predictive performances are as follows: Among the six models, we found that M8a had a lower predictive error in RMSE and MAE than M7a and M9b, and M8b also had a lower predictive error than M7b and M9b. However, the average RMSE (MAE) of M8a and M8b is equal to 1.7572 (0.9751) and 1.5965 (0.8743), respectively, which are larger than those in the WTI-related models (M3a, M4a) and (M3b, M4b), but smaller than those in the SPX-related models (M1a, M2a) and (M1b, M2b), and the SPX+WTI-related models (M5a, M5b) and (M6a, M6b), respectively. The integrated models were combined with the six SPX and WTI-related models with the random walk or AR(1) model. Since stock and oil prices are informative in forecasting the VIX, these integrated models generally have better forecasting ability than the random walk or AR(1) model, except for M7a through M9a at $\lambda = 1.0$. Additionally, because the WTI-related models generally have better forecast ability than those in SPX-related and SPX+WTI-related models, M8a and M8b also have better forecasting ability than those in M7a and M9a, as well as M7b and M9b, respectively. In other words, when some regression models have better predictive abilities, their integrated models also have better predictive abilities.

Discount Rate

We now analyze whether the adopted discount rate (λ) can affect the forecast ability of various regression models. The average RMSE of models M0a through M9a decreases from 1.0879 at λ = 1.0 to 0.9494 at λ = 0.40 and then increases to 0.9595 at λ = 0.30. This indicates that the forecast error in the RMSE is a U-shaped function of λ. In addition to the random walk and AR(1) models, all the forecast errors in RMSE and MAE are U-shaped functions of λ for the other 18 models (M1a through 9a, M1b through M9b). This implies that using the EWLS method to estimate the regression model generally has better forecasting ability than using the OLS method (λ = 1). Additionally, choosing an appropriate discount rate (λ) to properly distribute the weight of the old and new data can help improve the predictive ability of the regression model. This result is consistent with that of Wang et al. (2020), who found that models using TWLS estimation have stronger predictive power for forecasting stock returns than those using OLS estimation. Finally, among all the regression models, this study finds that M4b (with explanatory variables V_t and $\widehat{\Delta O}_{t+1}$) has the best forecast ability with an RMSE of 1.3826 at λ = 0.50 and an MAE of 0.6962 at λ = 0.60.

Investment Performance Comparison

Table 6 displays the investment performance of the mean returns of various regression models. Based on Subsection 3.5, the investment strategy is to buy (sell) if the one-day-ahead volatility forecast of the VIX price is larger (smaller) than the current VIX price. That is, the net investment position on the VIX is equal to 1, 0, and -1 if the next-day VIX price is expected to rise, remain unchanged, and fall, respectively. Accordingly, we obtain the average daily returns over the out-of-sample period for various regression models. Because the random walk model forecasts that the expected VIX price is unchanged, the corresponding investing position is zero, and its mean return is equal to zero. Intuitively, we assume that the average returns of various models are positively related to their predictive power. First, all average returns are positive for models M0a through M9a and M0b through M9b. This finding reveals that using stock and oil prices is informative for VIX investment. The average return of Models M0b through to M9b is equal to 0.0206, which is larger than 0.0173 of the average return of Models M0a through M9a. This result indicates that adding a one-day lag of the VIX into the regression models can improve average investment returns. Third, in addition to the random walk model, the average returns for the various models are hump-shaped with respect to the discount rate (λ). This reveals that the EWLS method outperforms the OLS method in VIX investment, and properly choosing the discount rate in various regression models can increase their average returns. Fourthly, the WTI-related models have larger average investment returns relative to the others, in which M3b, M4b, and M8b over $\lambda = 0.30$ through 1.0 have the three largest average returns of 0.0345, 0.0346, and 0.0345, respectively. In particular, the average return reached a peak of 0.0400 for M3b and M8b at $\lambda = 0.60$. In short, the average returns of various models are generally positively related to their predictive abilities.

Table 4 Out-of-the-Sample RMSE Comparison ($\mathrm{T} = 60$)

λ	*M0a*	*M1a*	*M2a*	*M3a*	*M4a*	*M5a*	*M6a*	*M7a*	*M8a*	*M9a*	*AVG*
1.00	1.9051	1.9233	1.9198	1.9285	1.9136	1.9221	1.9199	1.9062	1.9131	1.9095	1.9173
0.95	1.9051	1.9092	1.9059	1.8871	1.8730	1.9085	1.9067	1.8987	1.8932	1.8927	1.8972
0.90	1.9051	1.8958	1.8925	1.8395	1.8243	1.8976	1.8958	1.8902	1.8646	1.8701	1.8745
0.85	1.9051	1.8858	1.8825	1.7945	1.7782	1.8907	1.8889	1.8831	1.8358	1.8517	1.8546
0.80	1.9051	1.8783	1.8750	1.7532	1.7359	1.8865	1.8847	1.8782	1.8076	1.8330	1.8369
0.75	1.9051	1.8724	1.8691	1.7154	1.6974	1.8844	1.8826	1.8717	1.7804	1.8113	1.8205
0.70	1.9051	1.8680	1.8648	1.6818	1.6631	***1.8841***	***1.8823***	1.8668	1.7541	1.7894	1.8060
0.65	1.9051	1.8652	1.8620	1.6534	1.6338	1.8856	1.8840	1.8627	1.7288	1.7696	1.7939
0.60	1.9051	***1.8641***	***1.8610***	1.6314	1.6107	1.8891	1.8877	1.8594	1.7067	1.7523	1.7847
0.55	1.9051	1.8648	1.8619	1.6175	1.5955	1.8946	1.8936	1.8573	1.6880	1.7353	1.7787
0.50	1.9051	1.8676	1.8650	***1.6136***	***1.5904***	1.9027	1.9020	1.8557	1.6743	1.7225	***1.7771***
0.45	1.9051	1.8728	1.8707	1.6225	1.5981	1.9137	1.9134	***1.8551***	1.6684	1.7148	1.7810
0.40	1.9051	1.8812	1.8796	1.6470	1.6219	1.9287	1.9287	1.8568	***1.6663***	***1.7084***	1.7910
0.35	1.9051	1.8937	1.8928	1.6906	1.6653	1.9491	1.9495	1.8609	1.6784	1.7232	1.8115
0.30	1.9051	1.9116	1.9115	1.7569	1.7320	1.9768	1.9777	1.8678	1.6980	1.7341	1.8407
AVG	1.9051	1.8836	1.8809	1.7222	***1.7022***	1.9076	1.9065	1.8714	1.7572	1.7879	1.8244

λ	*M0b*	*M1b*	*M2b*	*M3b*	*M4b*	*M5b*	*M6b*	*M7b*	*M8b*	*M9b*	*AVG*
1.00	***2.2442***	1.8888	1.8813	1.8497	1.8328	1.8889	1.8825	1.8880	1.8624	1.8716	1.9090
0.95	2.3241	1.8740	1.8661	1.7513	1.7348	1.8773	1.8705	1.8780	1.8065	1.8348	1.8818
0.90	2.4112	1.8602	1.8519	1.6657	1.6471	1.8684	1.8613	1.8691	1.7506	1.7939	1.8580
0.85	2.5010	1.8490	1.8406	1.5920	1.5708	1.8624	1.8554	1.8629	1.6949	1.7545	1.8383
0.80	2.6017	1.8395	1.8313	1.5295	1.5061	1.8574	1.8508	1.8578	1.6428	1.7133	1.8230
0.75	2.7218	1.8328	1.8250	1.4796	1.4545	***1.8542***	***1.8480***	1.8512	1.5982	1.6751	1.8140
0.70	2.8669	***1.8299***	***1.8224***	1.4434	1.4169	1.8548	1.8491	1.8476	1.5595	1.6422	**1.8133**
0.65	3.0399	1.8303	1.8232	1.4209	1.3930	1.8602	1.8549	1.8461	1.5269	1.6135	1.8209
0.60	3.2402	1.8331	1.8263	***1.4118***	***1.3826***	1.8701	1.8654	***1.8459***	1.5012	1.5881	1.8365
0.55	3.4648	1.8374	1.8310	1.4159	1.3854	1.8841	1.8798	1.8468	1.4841	1.5658	1.8595
0.50	3.7104	1.8429	1.8368	1.4327	1.4010	1.9017	1.8977	1.8488	***1.4785***	1.5525	1.8903
0.45	3.9729	1.8496	1.8440	1.4617	1.4294	1.9225	1.9188	1.8517	1.4839	1.5501	1.9285
0.40	4.2477	1.8583	1.8531	1.5034	1.4715	1.9469	1.9433	1.8558	1.4963	***1.5479***	1.9724
0.35	4.5321	1.8699	1.8652	1.5599	1.5303	1.9764	1.9723	1.8618	1.5160	1.5534	2.0237
0.30	4.8290	1.8858	1.8814	1.6350	1.6098	2.0148	2.0082	1.8700	1.5453	1.5667	2.0846
AVG	3.2472	1.8521	1.8453	1.5435	***1.5178***	1.8960	1.8905	1.8588	1.5965	1.6549	1.8903

Notes: There are 4,279 daily data over the sample period from January 2, 2004 to December 31, 2020. However, there are only a total of 4,177 out-of-the-sample observations with moving window size T = 60 to be evaluated the forecast errors RMSEs ranging from May 28, 2004 to December 31, 2020. The discount rate λ corresponds to the EWLS (exponentially weighted least squares) methods. ***Bold italics*** numbers represent the smallest RMSEs among the competitive models.

Table 5 Out-of-the-Sample MAE Comparison ($T = 60$)

λ	***M0a***	***M1a***	***M2a***	***M3a***	***M4a***	***M5a***	***M6a***	***M7a***	***M8a***	***M9a***	***AVG***
1.00	1.0797	1.0964	1.0946	1.0825	1.0805	1.0962	1.0952	1.0839	1.0797	1.0818	1.0879
0.95	1.0797	1.0838	1.0831	1.0569	1.0559	1.0840	1.0841	1.0786	1.0668	1.0718	1.0739
0.90	1.0797	1.0708	1.0701	1.0272	1.0260	1.0723	1.0724	1.0719	1.0500	1.0580	1.0576
0.85	1.0797	1.0594	1.0584	0.9978	0.9964	1.0629	1.0631	1.0649	1.0333	1.0453	1.0424
0.80	1.0797	1.0485	1.0473	0.9697	0.9683	1.0549	1.0552	1.0581	1.0171	1.0318	1.0279
0.75	1.0797	1.0379	1.0368	0.9432	0.9417	1.0485	1.0486	1.0515	1.0009	1.0177	1.0141
0.70	1.0797	1.0276	1.0265	0.9177	0.9166	1.0432	1.0435	1.0448	0.9845	1.0028	1.0008
0.65	1.0797	1.0180	1.0171	0.8940	0.8932	1.0392	1.0397	1.0380	0.9680	0.9883	0.9884
0.60	1.0797	1.0092	1.0083	0.8728	0.8721	1.0365	1.0369	1.0314	0.9524	0.9740	0.9771
0.55	1.0797	1.0010	1.0001	0.8535	0.8527	***1.0354***	***1.0356***	1.0254	0.9378	0.9593	0.9667
0.50	1.0797	0.9939	0.9931	0.8374	0.8363	1.0364	1.0366	1.0197	0.9243	0.9463	0.9582
0.45	1.0797	0.9881	0.9874	0.8268	0.8239	1.0397	1.0400	1.0145	0.9127	0.9358	0.9521
0.40	1.0797	0.9848	0.9839	0.8239	***0.8203***	1.0459	1.0460	1.0106	0.9025	0.9269	***0.9494***
0.35	1.0797	***0.9845***	***0.9837***	***0.8300***	0.8263	1.0558	1.0558	1.0086	***0.8974***	0.9226	0.9516
0.30	1.0797	0.9877	0.9871	0.8471	0.8437	1.0705	1.0704	***1.0081***	0.8990	***0.9216***	0.9595
AVG	1.0797	1.0261	1.0252	0.9187	***0.9169***	1.0548	1.0549	1.0407	0.9751	0.9923	1.0005

λ	***M0b***	***M1b***	***M2b***	***M3b***	***M4b***	***M5b***	***M6b***	***M7b***	***M8b***	***M9b***	***AVG***
1.00	1.2235	1.0876	1.0851	1.0777	1.0766	1.0880	1.0863	1.0771	1.0670	1.0709	1.0940
0.95	1.2085	1.0719	1.0698	1.0004	1.0004	1.0736	1.0727	1.0695	1.0310	1.0462	1.0644
0.90	1.1969	1.0548	1.0528	0.9399	0.9387	1.0582	1.0574	1.0603	0.9943	1.0189	1.0372
0.85	***1.1910***	1.0374	1.0355	0.8868	0.8843	1.0434	1.0427	1.0504	0.9570	0.9896	1.0118
0.80	1.1912	1.0217	1.0197	0.8386	0.8358	1.0310	1.0301	1.0412	0.9211	0.9576	0.9888
0.75	1.1977	1.0089	1.0069	0.7980	0.7948	1.0229	1.0219	1.0328	0.8892	0.9294	0.9702
0.70	1.2125	0.9988	0.9969	0.7636	0.7598	1.0186	1.0177	1.0255	0.8608	0.9032	0.9557
0.65	1.2358	0.9903	0.9884	0.7368	0.7332	***1.0178***	***1.0169***	1.0192	0.8342	0.8780	0.9451
0.60	1.2703	0.9832	0.9814	0.7173	0.7136	1.0200	1.0193	1.0136	0.8122	0.8572	0.9388
0.55	1.3139	0.9775	0.9757	0.7047	0.7010	1.0245	1.0237	1.0092	0.7957	0.8403	***0.9366***
0.50	1.3664	0.9735	0.9719	***0.6999***	***0.6962***	1.0316	1.0304	1.0060	0.7862	0.8286	0.9391
0.45	1.4274	0.9712	0.9697	0.7036	0.6995	1.0413	1.0400	1.0037	***0.7827***	0.8235	0.9463
0.40	1.5019	***0.9708***	***0.9693***	0.7144	0.7103	1.0534	1.0520	***1.0021***	0.7850	***0.8199***	0.9579
0.35	1.5967	0.9730	0.9716	0.7337	0.7294	1.0690	1.0674	1.0022	0.7927	0.8238	0.9759
0.30	1.7106	0.9786	0.9772	0.7630	0.7590	1.0889	1.0868	1.0046	0.8057	0.8333	1.0008
AVG	1.3230	1.0066	1.0048	0.8052	***0.8022***	1.0455	1.0444	1.0278	0.8743	0.9080	0.9842

Notes: There are 4,279 daily data over the sample period from January 2, 2004 to December 31, 2020. However, there are only a total of 4,177 out-of-the-sample observations with moving window size T = 60 to be evaluated the forecast errors MAEs ranging from May 28, 2004 to December 31, 2020. The discount rate λ corresponds to the EWLS (exponentially weighted least squares) methods. ***Bold italics*** numbers represent the smallest MAEs among the competitive models.

Table 6 Out-of-the-Sample Average Investment Return Comparison ($T = 60$)

λ	M0a	M1a	M2a	M3a	M4a	M5a	M6a	M7a	M8a	M9a	AVG
1.00	0.0000	0.0004	0.0010	0.0071	0.0067	0.0002	0.0009	0.0010	0.0069	0.0048	0.0032
0.95	0.0000	0.0030	0.0037	0.0159	0.0157	0.0031	0.0035	0.0036	0.0161	0.0135	0.0087
0.90	0.0000	0.0072	0.0076	0.0210	0.0216	0.0069	0.0070	0.0073	0.0212	0.0185	0.0131
0.85	0.0000	0.0090	0.0093	0.0254	0.0255	0.0088	0.0088	0.0089	0.0256	0.0215	0.0159
0.80	0.0000	0.0108	0.0108	0.0269	0.0270	0.0094	0.0098	0.0108	0.0267	0.0243	0.0174
0.75	0.0000	0.0118	0.0121	0.0286	0.0285	0.0103	0.0102	0.0120	0.0286	0.0257	0.0186
0.70	0.0000	0.0130	0.0130	0.0299	0.0293	0.0106	0.0104	0.0131	0.0297	0.0269	0.0195
0.65	0.0000	0.0134	0.0135	0.0307	0.0304	***0.0107***	0.0106	0.0134	0.0305	0.0269	0.0200
0.60	0.0000	0.0142	0.0145	***0.0320***	0.0310	0.0105	***0.0110***	0.0142	***0.0316***	0.0273	0.0207
0.55	0.0000	0.0145	0.0146	0.0315	***0.0314***	0.0107	0.0105	0.0147	0.0314	***0.0275***	0.0208
0.50	0.0000	0.0146	0.0148	0.0313	0.0314	0.0104	0.0105	0.0147	0.0312	0.0275	0.0207
0.45	0.0000	***0.0153***	***0.0155***	0.0309	0.0312	0.0102	0.0104	***0.0153***	0.0309	0.0273	***0.0208***
0.40	0.0000	0.0153	0.0152	0.0302	0.0300	0.0100	0.0103	0.0152	0.0300	0.0273	0.0204
0.35	0.0000	0.0150	0.0154	0.0296	0.0301	0.0103	0.0104	0.0149	0.0298	0.0263	0.0202
0.30	0.0000	0.0151	0.0149	0.0287	0.0293	0.0103	0.0102	0.0151	0.0289	0.0254	0.0197
AVG	0.0000	0.0115	0.0117	***0.0266***	***0.0266***	0.0088	0.0090	0.0116	***0.0266***	0.0234	0.0173

λ	M0b	M1b	M2b	M3b	M4b	M5b	M6b	M7b	M8b	M9b	AVG
1.00	-0.0227	0.0072	0.0072	0.0119	0.0126	0.0078	0.0073	0.0071	0.0122	0.0119	0.0062
0.95	-0.0144	0.0095	0.0106	0.0232	0.0235	0.0099	0.0105	0.0096	0.0233	0.0203	0.0126
0.90	-0.0059	0.0122	0.0122	0.0295	0.0297	0.0123	0.0121	0.0122	0.0297	0.0259	0.0170
0.85	-0.0016	0.0139	0.0143	0.0337	0.0336	0.0128	0.0129	0.0141	0.0336	0.0296	0.0197
0.80	0.0013	0.0149	0.0154	0.0369	0.0369	0.0135	0.0137	0.0152	0.0368	0.0321	0.0217
0.75	0.0037	0.0159	0.0162	0.0379	0.0383	0.0140	0.0135	0.0162	0.0381	0.0336	0.0227
0.70	0.0049	***0.0167***	0.0170	0.0387	0.0387	0.0137	0.0139	0.0167	0.0388	0.0345	0.0234
0.65	0.0070	0.0164	0.0167	0.0392	0.0394	***0.0140***	0.0139	0.0164	0.0392	0.0352	0.0238
0.60	0.0087	0.0160	0.0165	***0.0400***	***0.0399***	0.0135	0.0136	0.0162	***0.0400***	***0.0354***	0.0240
0.55	0.0093	0.0165	0.0169	0.0395	0.0396	0.0134	0.0136	***0.0168***	0.0395	0.0351	***0.0240***
0.50	0.0096	0.0166	***0.0172***	0.0391	0.0388	0.0135	***0.0140***	0.0167	0.0390	0.0341	0.0239
0.45	0.0101	0.0165	0.0168	0.0383	0.0383	0.0129	0.0137	0.0166	0.0382	0.0336	0.0235
0.40	***0.0102***	0.0164	0.0163	0.0375	0.0373	0.0125	0.0130	0.0164	0.0373	0.0332	0.0230
0.35	0.0101	0.0161	0.0164	0.0364	0.0362	0.0124	0.0125	0.0162	0.0365	0.0322	0.0225
0.30	0.0096	0.0155	0.0157	0.0352	0.0355	0.0118	0.0122	0.0156	0.0354	0.0313	0.0218
AVG	0.0027	0.0147	0.0150	0.0345	***0.0346***	0.0125	0.0127	0.0148	0.0345	0.0305	0.0206

Notes: There are 4,279 daily data over the sample period from January 2, 2004 to December 31, 2020. However, there are only a total of 4,177 out-of-the-sample observations with moving window size T = 60 to be evaluated the forecast errors RMSEs ranging from May 28, 2004 to December 31, 2020. The discount rate λ corresponds to the EWLS (exponentially weighted least squares) methods. ***Bold italics*** numbers represent the smallest RMSEs among the competitive models.

4.3 Robust Examination

This subsection aims to examine whether the main results in Subsection 4.2 can remain unchanged with respect to different moving window sizes. Similar to Tables 4 through 6 for a moving window size of T = 60, this study additionally completes 12 tables corresponding to T = 20, 40, 80, and 100.[5] Regardless of the moving window size, the forecast error in RMSE and MAE for various regression models generally has a U-shaped function of the discount rate, indicating that properly choosing the discount rate can improve their forecast abilities. Furthermore, to compare whether the adopted moving window size affected the predictive power of various regression models, we averaged the predictive performance of each model over 15 different values of λ (0.30, 0.35, …, 1.00). The results are presented in Tables 7–9.

Table 7 reports the RMSE and MAE comparison against different moving window sizes (T) for various regression models. Firstly, on average, the forecast errors in RMSE and MAE decrease slowly with increasing T. However, their differences are very small. Adding a one-day lag of VIX into the regression models (M0b through M9b) can decrease the forecast errors. That is, the average RMSE and MAE in M1b through M9b are lower than those in M1a through M9b, respectively. Thirdly, M4b has the smallest average RMSE (1.5189) and MAE (0.8024) among all regression models. Indeed, M4b has the smallest RMSE and MAE for all moving window sizes. Particularly, M4b at T = 20 and T = 40 have the smallest RMSE and MAE, respectively. Subsequently, Table 8 displays the investment performance in average return for various regression models. We find that those models with better forecast ability also have larger average investment returns. In short, M4b (with explanatory variables V_t and $\widehat{\Delta O}_{t+1}$) has the best forecast ability and average investment return regardless of moving window size. Additionally, besides the random walk (M0a) and AR(1) (M0b) models, this study found that adding a one-day lag of VIX into the regression models can improve their forecast ability and average investment returns, whatever the moving window size. Moreover, the forecast ability of each regression model does not necessarily increase or decrease monotonically with moving window size.

[5] Due to space limitations, these tables are omitted here. Interested readers can request from the authors.

Table 7 Forecast Ability Comparison Against Moving Window Size (T)

Panel A. RMSE

T	*M0a*	*M1a*	*M2a*	*M3a*	*M4a*	*M5a*	*M6a*	*M7a*	*M8a*	*M9a*	*AVG*
20	1.9051	1.9397	1.9341	1.7370	1.7169	1.9625	1.9615	1.8991	1.7685	1.8078	1.8632
40	1.9051	1.8985	1.8951	1.7271	1.7060	1.9228	1.9218	1.8795	1.7596	1.7930	1.8409
60	1.9051	1.8836	1.8809	1.7222	1.7022	1.9076	1.9065	1.8714	***1.7572***	1.7879	1.8325
80	1.9051	1.8751	1.8725	***1.7167***	***1.6983***	1.8997	1.8983	1.8674	***1.7572***	***1.7856***	1.8276
100	1.9051	***1.8703***	***1.8677***	1.7192	1.7022	***1.8955***	***1.8920***	***1.8654***	1.7592	1.7860	**1.8263**
AVG	1.9051	1.8934	1.8901	1.7245	**1.7051**	1.9176	1.9160	1.8766	1.7603	1.7920	1.8381

T	*M0b*	*M1b*	*M2b*	*M3b*	*M4b*	*M5b*	*M6b*	*M7b*	*M8b*	*M9b*	*AVG*
20	3.2660	1.9106	1.9022	1.5401	1.5143	1.9674	1.9616	1.8885	***1.5919***	1.6617	1.9204
40	3.2500	1.8695	1.8621	1.5448	1.5176	1.9211	1.9149	1.8694	1.5950	1.6552	1.8999
60	3.2472	1.8521	1.8453	1.5435	1.5178	1.8960	1.8905	1.8588	1.5965	***1.6549***	1.8903
80	3.2458	1.8435	1.8367	***1.5424***	***1.5186***	1.8866	1.8808	1.8530	1.5992	1.6554	***1.8862***
100	***3.2450***	***1.8397***	***1.8325***	1.5498	1.5264	***1.8818***	***1.8739***	***1.8516***	1.6035	1.6581	***1.8862***
AVG	3.2508	1.8631	1.8558	1.5441	***1.5189***	1.9106	1.9043	1.8642	1.5972	1.6571	1.8966

Panel B. MAE

T	*M0a*	*M1a*	*M2a*	*M3a*	*M4a*	*M5a*	*M6a*	*M7a*	*M8a*	*M9a*	*AVG*
20	1.0797	1.0618	1.0611	0.9261	0.9252	1.0904	1.0912	1.0572	0.9775	1.0003	1.0271
40	1.0797	1.0349	1.0338	0.9204	0.9182	1.0637	1.0636	1.0446	***0.9748***	0.9936	1.0127
60	1.0797	1.0261	1.0252	0.9187	0.9169	1.0548	1.0549	1.0407	0.9751	0.9923	1.0084
80	1.0797	1.0217	1.0207	***0.9178***	***0.9158***	1.0513	1.0508	1.0377	***0.9748***	0.9907	1.0061
100	1.0797	***1.0173***	***1.0162***	0.9183	0.9166	***1.0484***	***1.0467***	***1.0355***	0.9749	***0.9896***	***1.0043***
AVG	1.0797	1.0324	1.0314	0.9202	***0.9185***	1.0617	1.0614	1.0431	0.9754	0.9933	1.0117

T	*M0b*	*M1b*	*M2b*	*M3b*	*M4b*	*M5b*	*M6b*	*M7b*	*M8b*	*M9b*	*AVG*
20	1.3234	1.0357	1.0339	0.8057	0.8027	1.0764	1.0758	1.0398	***0.8717***	0.9105	0.9976
40	***1.3225***	1.0133	1.0113	***0.8040***	***0.8005***	1.0540	1.0526	1.0310	***0.8717***	***0.9072***	0.9868
60	1.3230	1.0066	1.0048	0.8052	0.8022	1.0455	1.0444	1.0278	0.8743	0.9080	0.9842
80	1.3228	1.0032	1.0012	0.8061	0.8027	1.0414	1.0396	1.0246	0.8750	0.9075	0.9824
100	1.3227	***0.9996***	***0.9975***	0.8072	0.8038	***1.0386***	***1.0360***	***1.0227***	0.8758	0.9078	***0.9812***
AVG	1.3229	1.0117	1.0097	0.8056	***0.8024***	1.0512	1.0497	1.0292	0.8737	0.9082	0.9864

Notes: There are 4,279 daily data over the sample period from January 2, 2004 to December 31, 2020. However, there are only a total of 4,177 out-of-the-sample observations T = 60 to be evaluated the forecast errors RMSEs and MAEs ranging from May 28, 2004 to December 31, 2020. The RMSEs and MAEs are averaged from each regression mode over 15 different discount values of λ (0.30, 0.35, …, 1.00) corresponding to the EWLS (exponentially weighted least squares) method. ***Bold italics*** numbers represent the smallest RMSEs or MAEs among the competitive models.

Table 8 Average Investment Return Comparison against Moving Window Size (T)

T	*M0a*	*M1a*	*M2a*	*M3a*	*M4a*	*M5a*	*M6a*	*M7a*	*M8a*	*M9a*	*AVG*
20	0.0000	0.0090	0.0090	0.0257	0.0258	0.0060	0.0060	0.0090	0.0257	0.0206	0.0137
40	0.0000	0.0108	0.0108	***0.0266***	***0.0266***	0.0079	0.0079	0.0108	***0.0266***	0.0230	0.0151
60	0.0000	0.0115	0.0117	***0.0266***	***0.0266***	***0.0088***	***0.0090***	0.0116	***0.0266***	0.0234	0.0156
80	0.0000	***0.0126***	***0.0127***	***0.0264***	0.0265	0.0104	0.0106	***0.0127***	0.0265	***0.0239***	***0.0162***
100	0.0000	0.0111	0.0113	***0.0264***	0.0264	0.0086	0.0087	0.0112	0.0264	0.0229	0.0153
AVG	0.0000	0.0110	0.0111	0.0263	***0.0264***	0.0083	0.0084	0.0111	0.0264	0.0228	0.0152

T	*M0b*	*M1b*	*M2b*	*M3b*	*M4b*	*M5b*	*M6b*	*M7b*	*M8b*	*M9b*	*AVG*
20	***0.0041***	0.0141	0.0143	0.0346	***0.0348***	0.0114	0.0116	0.0142	0.0347	0.0300	0.0204
40	0.0032	0.0147	0.0149	***0.0347***	***0.0348***	0.0119	0.0122	0.0148	***0.0348***	***0.0309***	***0.0207***
60	0.0027	0.0147	0.0150	0.0345	0.0346	***0.0125***	***0.0127***	0.0148	0.0345	0.0305	0.0206
80	0.0027	0.0147	0.0150	0.0345	0.0346	***0.0125***	***0.0127***	0.0148	0.0345	0.0305	0.0206
100	0.0029	***0.0149***	***0.0151***	0.0345	0.0346	0.0124	***0.0127***	***0.0150***	0.0345	0.0304	***0.0207***
AVG	0.0031	0.0146	0.0149	0.0345	***0.0347***	0.0122	0.0124	0.0147	0.0346	0.0305	0.0206

Notes: There are 4,279 daily data over the sample period from January 2, 2004 to December 31, 2020. However, there are only a total of 4,177 out-of-the-sample observations to be evaluated the average investment returns ranging from May 28, 2004 to December 31, 2020. The average investment returns are averaged from each regression mode over 15 different discount values of λ (0.30, 0.35, …, 1.00) corresponding to the EWLS (exponentially weighted least squares) method. ***Bold italics*** numbers represent the largest average investment returns among the competitive models.

5. Conclusions

This study forecasted the VIX based on stock and crude oil prices, which were represented by SPX and WTI, respectively. Although many studies have predicted the VIX, few studies have considered both stock and oil prices in the prediction. We considered 20 regression models to predict VIX, with explanatory variables including the one-day lag of VIX, SPX, and WTI. These regression models are estimated using the exponentially weighted least squares (EWLS) method, in which newer observations are assigned more weight than older ones. Using daily observations from 2004 to 2020 as the sample, the main findings of this study are as follows.

First, previous VIX information has a considerable predictive ability for forecasting the future VIX. Second, using the stock or oil price variables alone helps predict the VIX, but using both reduces the predictive power. In particular, using the oil price is more informative than using the stock price in forecasting the VIX. Third, the forecast errors (RMSE and MAE) of each regression model are generally a U-shaped function of the discount rate in the EWLS method, whereas their forecast ability does not necessarily increase or decrease monotonically with the moving window size. Fourth, the average returns of various VIX investment models are positively related to their predictive power. Fifth, integrated models usually have a desirable prediction performance in forecasting the VIX. Finally, the regression model with explanatory variables of a one-day lag of the VIX and expected change in oil price generally has the best forecast ability and investment performance in the VIX.

Relative to the random walk and AR(1) models, this study demonstrated that adding stock and oil price information into the regression models can substantially improve the VIX forecast ability. Since previous studies have shown that several regression models (such as mean-reverting, BN-S, HAR, and ARMA-GARCH models) can accurately predict the VIX, future research can investigate whether adding stock and oil price information into these models can improve their forecast ability. In addition to RMSE, MAE, and average return, future work can also take the model confidence set approach to compare the predictive performance among various regression models. Moreover, since VIX has a long-memory property, besides the one-day lag of VIX, future work can consider adopting the weekly and monthly lagged average VIX values into the regression models to forecast VIX.

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