JEL classification: G21, C70, D85.
Keywords: interbank market; liquidity networks; deposit withdrawals; cash holdings; imperfectly negatively correlated shocks

Liquidity Networks in Banking

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Abstract

Modern financial and banking systems are very much interconnected. In a setting where banks are prone to liquidity risk due to early withdrawals by depositors, this paper analyzes the optimal liquidity network relationship that banks will settle. The paper interprets the network relationship as the exchange of 'committed credit lines' contracts between the banks. The paper shows that the given liquidity network of Allen and Gale (2000) is one of the optimal solutions that may occur and a risk-based pricing takes place in the interbank market. Banks dispose of their liquidity risk and reduce the total required cash holdings of the banking system to cover early withdrawals by means of this relationship. Additionally, the paper considers the case where liquidity shocks of banks become imperfectly negatively correlated. The network relationship between banks under imperfectly negatively correlated shocks is even robust to the extreme case, in which there is no reduction in the total required cash holdings of banks.

1. Introduction

In today's world, financial systems are highly interconnected, as seen during the subprime mortgage crisis. This has attracted a lot of attention of researchers, since these linkages can lead to the collapse of an entire system through contagious failures. Previous research mostly tried to understand whether such a collapse can really occur and how stable the system is when such inter-linkages exist between financial institutions. This study tries to understand the incentives behind such inter-linkages and to determine the network relationship between two financial institutions, say banks in separate regions.

This paper studies a setting where banks are prone to liquidity risk due to early withdrawals by depositors. In this setting, the paper determines the optimal liquidity network relationship or financial contract that will occur between two banks. This network relationship may be understood as 'committed credit lines' between the banks. Thus, banks become insurers of each other against the liquidity shock in hedgeable states. By means of this network relationship, they are able to completely get rid of the liquidity risk and reduce the total required cash holdings of the whole banking system when their liquidity shocks are perfectly negatively correlated. However, it is not possible for banks to reduce the variance of the liquidity shock to zero if their shocks become imperfectly negatively correlated. Interestingly, the liquidity network relationship between banks with imperfectly negatively correlated shocks is even robust to the extreme case: the case in which the relationship does not induce any reduction in the total required cash holdings of banks to cover early withdrawals. The reason is that banks are still able to agree on a price or (net) upfront commitment fee for the credit lines contract that leaves both of them indifferent between staying independent and settling a network. Yet, the set of feasible prices gets narrowed with less negatively correlated shocks.

As already stated, earlier theoretical literature has focused more on the effects of exogenously given network structures on contagious failures instead of its formation and incentives behind its formation. However, there are also some studies looking at endogenously determined intermediary (deposit) contracts starting with Briant (1980) and Diamond and Dybvig (1983). These papers study inter-temporal models where depositors face privately observed preference shocks and time to maturity determines returns to investment. With such preference shocks, they show the fact that deposit contracts are superior to Walrasian trading solutions to provide insurance for agents. Following these two papers, Bhattacharya and Gale (1987) change the setup slightly and consider the situation where banks face privately observed shocks that determines the proportion of early withdrawals. They show that the optimal mechanism design in this case is subject to second-best distortions even when there is no aggregate liquidity shock in the system. These second best distortions occur due to the fact that banks have an incentive to underinvest in the liquid asset since they can free-ride on the interbank market liquidity pool. Taking this setup one step further, Allen and Gale (2000) look at whether financial contagion could occur when banks insure themselves against liquidity shocks with an interbank market. The authors test this question with exogenously selected network structures: complete vs. incomplete. They conclude that complete networks in which each region is connected to all the other regions are more resilient to contagion than incomplete structures.

Following Allen and Gale (2000), several theoretical papers considered the optimal network design and the threat of contagion as a result of it. Especially, Leitner (2005) argues that under such a contagion threat, it may be optimal for some agents to bail out the others to prevent the collapse of the whole network. Going one step forward, Babus (2013) shows that such a network relationship where banks bail out each other comes out as an equilibrium solution in a network formation game. This paper also shares a similar idea that is banks may act as insurers of each other by exchanging 'committed credit lines' contracts.

Several empirical studies have also tried to find evidence of contagious failures in different national banking systems. With this aim, these studies have tried to quantify the size of contagion by identifying the mutual claims banks have on each other. In general, such papers use balance sheet data in order to estimate bilateral credit relationships. They test the stability of the whole banking system by simulating failure of a single bank like Degryse and Nguyen (2007), Sheldon and Maurer (1998) and Elsinger, Lehar and Summer (2006).

Instead of looking for contagious effects resulting from different types of network relationships within banks, this paper tries to endogenously determine the optimal liquidity network relationship that banks will settle and to understand the characteristics of such a relationship. Within the cited theoretical papers above, this paper is closest to Bhattacharya and Gale (1987) and Allen and Gale (2000). The main difference of the current paper from Bhattacharya and Gale (1987) is to study an interbank coordination problem when the return on the risky asset depends on the early liquidation decisions by both banks. Such a return structure represents common investments in the asset portfolio of the banks and early liquidation/sale of the asset by other banks would affect the price of the asset as it happened during the subprime mortgage crisis. In this situation, the paper shows that the problem of underinvestment in the liquid asset and free riding by banks which is the main result of Bhattacharya and Gale (1987) is resolved. Following Allen and Gale (2000), the model adopts a framework where consumers have random liquidity needs as in Diamond and Dybvig (1983), which creates late versus early consumers. Despite having a similar framework regarding the consumer preferences, this paper has several differences regarding other important elements. First, while in the above papers investment portfolio decisions of banks are determined as the solution to a social planner's problem, here banks decide on their investment portfolios to maximize profits. Second, Allen and Gale (2000) restrict the network relationship of banks to the case of deposits exchange with the assumption that each bank receives the same return as the consumers for the deposits transferred in the interbank market. This paper, however, lets the network relationship to be more general by interpreting it as the 'committed credit lines' between the banks and determines its parameters in equilibrium.

With this setup, the paper shows that the liquidity network of Allen and Gale (2000) is one of the optimal solutions that may occur. The network relationship of this paper becomes equivalent to the one in Allen and Gale (2000) if the credit lines between the banks are set to be symmetric or if the holdings are set to be symmetric across banks, to better say it in the original paper's terms. However, symmetry of the credit lines or the holdings is not a must, since the (net) upfront commitment fee will adjust accordingly depending on the credit lines contract, either symmetric or non-symmetric. Although it is beyond the scope of this paper, since the symmetry of holdings is an important determinant of contagious effects as it is stated in Allen and Gale (2000), non-symmetric holdings may lead to different results from the viewpoint of contagion in the system. In any case, the banks perfectly insure themselves against liquidity shocks with this network relationship as in Allen and Gale (2000). Moreover, the characteristics of the optimal network relationship illustrate that there is risk-based pricing in the interbank market which is supported by recent empirical evidence despite the traditional view of single pricing in the interbank market (King, 2008, Ho and Saunders, 1985). It is seen that the (net) upfront commitment fee or the price of the credit lines contract adjusts with respect to the idiosyncratic risks of the banks. In the final section, the paper checks the robustness of this liquidity network relationship by considering alternatively the case of imperfectly negatively correlated liquidity shocks. Interestingly, the network relationship between banks is found to be robust even when they face an aggregate liquidity shock in one of the states.

This paper is organized as follows. Section 2 introduces the model with perfectly negatively correlated liquidity shocks. Section 3 includes the solution of the model which determines investment portfolio decisions of banks and the optimal network relationship. Section 4 considers the case of imperfectly negatively correlated shocks. Finally, Section 5 concludes.

2. Model Description

This section describes a model to capture the idea that banks confronted with negatively correlated liquidity shocks may prefer to enter a financial relationship with each other. The model has three time periods t_0, t_1, t_2 and two separate regions. In each region, there is one bank. Thus, one may perceive banks and regions as equivalent. At t_0 each bank obtains one unit of deposits from consumers. All consumers are identical as of t_0. At t_0, each consumer may face a privately observed liquidity shock that creates a necessity for the consumer to withdraw the deposit

immediately. The two regions differ with respect to the ratio of this type of consumers that will be referred to as 'early consumers' throughout
the paper. The proportion of early consumers may be either $a_H$or $a_L$in each region where $a_L < a_H$. Accordingly, there exist two different states ($\omega \epsilon\{1,2\}$) with respect to the proportion of early consumers in each region. In state $\omega = 1$, which occurs with probability p_1, the proportion of early consumers is a_H in region 1 (R_1) and a_L in region 2 (R_2). On the other hand, in state $\omega = 2$, which realizes with probability $p_2 = 1 - p_1$, the proportion of early consumers becomes a_L and a_H in region 1 (R_1) and region 2 (R_2), respectively. This may be represented as follows:

$$\text{Theproportionofearlyconsumers} =: \begin{Bmatrix} \mathrm{a_H inR_1, a_L inR_2 withp_1} \\ \mathrm{a_L inR_1, a_H inR_2 withp_2} \end{Bmatrix} \quad (1)$$

This negative correlation among early consumers creates an incentive for banks to initiate a financial network. Early consumers, who withdraw at t_1, obtain a return of $c_1 > 1$ from their deposits. The remaining consumers, who wait and do not withdraw until t_2, obtain a higher return c_2 from their deposits such that $c_2 > c_1$. Here the variables c_i are assumed to be exogenous in order to focus on the optimal network relationship between banks.

On the asset side, banks may invest in a common project and a riskless asset. The riskless asset has a return of 1 and may be considered as cash. From now on, the riskless asset will be referred to as cash in the paper. Into the common project both banks invest an amount of their choice. The return on the common project $r\epsilon\{\tilde{r}, \underline{r}, \overline{r}\}$ is dependent upon whether the banks liquidate their shares in the project at t_1 or not. The return on this project is $\bar{r} > 1$ if both banks keep their shares in the project until the economy resolves at t_2. Due to early withdrawals, it could be that banks need to liquidate their share in the project at t_1. The bank that liquidates its share in the project obtains a return of $\tilde{r} < 1$ and pays back all consumers at t_1. However, in the case of liquidation at t_1 with a return of $\tilde{r} < 1$, the bank goes into bankruptcy and obtains zero payoff due to its limited liability. In such a situation, the other bank that did not liquidate its share is also affected by this liquidation and obtains a lower return $\underline{r}$, where $\tilde{r} < 1 < \underline{r} < \overline{r}$. In the case of liquidation by both banks, both of their returns on the project are $\tilde{r}$ and they both declare bankruptcy. This return structure aims to capture the idea of a bank run on the asset side due to the common bad performing investments as observed in the subprime mortgage market.

This paper looks at what kind of liquidity network relationship (or financial contract) is optimal between banks in such a scenario. Banks face a tradeoff: On the one hand, they would like to invest as much as possible of the deposits in the common project. On the other hand, they need to keep enough cash at hand in order to cover the withdrawals that occur at t_1 due to early consumers. It will be seen that banks prefer to insure each other in order to optimize cash holdings and prevent the losses associated with early liquidation.

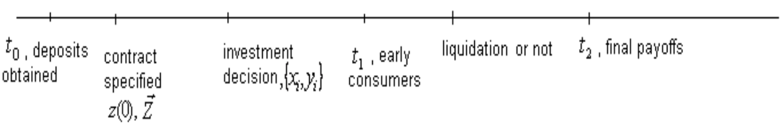
The aim is to determine the optimal network relationship or the optimal financial contract that takes place between banks. This relationship may be interpreted as the exchange of 'committed credit lines' contracts between banks to bilaterally insure themselves against the liquidity risk that they face at t_1. The credit amounts exchanged by banks resulting from the contract at t_1 are denoted by $\vec{z}= (z(1), z(2))$. Thus the net credit amount they exchange is $z(1)$ in state $\omega = 1$ and $z(2)$ in state $\omega = 2$.

The (net) upfront commitment fee or the price of this 'committed credit lines' contract at t_0 is denoted by $z(0)$. Such upfront commitment fees are common for 'committed credit lines' contracts and also formalized by earlier theoretical literature (Boot, Thakor and Udel (1987)).

Alternatively, if the network relationship between banks is interpreted as a bilateral insurance contract, one may perceive the price $z(0)$ as the net of insurance premia settled at t_0 and the payoff vector $\vec{z}$ as the insurance payments taking place at t_1 . Then, the task is to determine the optimal $z(0)$and $\vec{z}$.

After banks specify the optimal contract ($z(0)$ and $\vec{z}$) and $z(0)$ clears between banks, they also determine their investment decisions at t_0: how much to invest into the common project and how much in the riskless asset. Accordingly, x_i denotes bank i's investment in the riskless asset and y_i denotes bank i's investment in the common project where $i \epsilon \{1,2\}$. Then, the bank which has a long position (buyer of the 'committed credit' lines contract) owns an amount of $1 - z(0)$ for investment purposes at t_0. The other bank, having a short position (seller of the contract), has an amount of $1 + z(0)$ to invest in cash and the common project. It does not actually matter which one of the banks longs or shorts the asset since the price of the contract $z(0)$ is solely determined in equilibrium so that both banks are at least as good as with the financial relationship compared to the situation where they are stand-alone. Suppose that bank 1 buys and bank 2 sells the contract for the rest of the paper. Then, the following budget equations hold for bank 1 and bank 2, respectively: $x_1 + y_1 + z(0) = 1$ and $x_2 + y_2 - z(0) = 1$. To summarize, Figure 1 below provides a timeline for the game.

Figure 1 Timeline of Events



Notes: This figure depicts how the sum of banks' expected payoffs changes with respect to the difference of net credit amounts exchanged by banks (or the difference of insurance payments in states 1 and 2). The payoff function strictly increases under 'Case 1' that occurs when z(1)-z(2) is smaller than ($a_H c_1 - a_L c_1$) and after that point it starts to decrease when z(1)-z(2) is bigger than ($a_H c_1 - a_L c_1$). Thus the optimal point for the total payoff function occurs at z(1)-z(2)= ($a_H c_1 - a_L c_1$) where 'Case 1' and 'Case 2' overlaps. In other words, banks match the necessary cash amounts of the two states to prevent early liquidation with the committed credit lines.

This game is a game where banks play against the Nature with simultaneous moves. We will apply here the solution method of 'Backwards Induction'. In other words, we will first deal with the investment portfolios of banks and then go back in time and determine the optimal network relationship between banks.

Throughout the paper, we will assume that banks are infinitely loss-averse.

Thus, the banks have the following utility function:

$$U_i(\Pi_{i,t_2}) = \Pi_{i,t_2} \tag{2}$$

where $\Pi_i \ t_2$ is the realized profit of each bank at t_2. Additionally, we disregard discounting between time periods.

The source of the liquidity risk in the model is clearly the early withdrawals at t_1. In each state, banks face a different level of early withdrawals. That means the required cash amount to prevent an early liquidation and so bankruptcy also varies between states. The required cash amounts or the threshold cash levels to prevent a bankruptcy depend on the payoff vector $\vec{z}$. Payoff vector $\vec{z}$ settles between banks at t_1 just before the early withdrawals occur. While the payoff vector enters into bank 1's account with a positive sign (long position), it enters into bank 2's account with a negative sign (short position). Then, bank 1 needs a cash amount equal to early withdrawals minus the payoff from the contract in that state not to bankrupt: Accordingly, threshold cash levels for bank 1 in states $\omega = 1$ and $\omega = 2$ are $a_H c_1 - z(1)$ (i) and $a_L c_1 - z(2)$ (ii), respectively. Which one of the threshold cash levels of banks is larger depends on the payoff vector $\vec{z}$. Realize that $(i) \geq (ii)$ and $(iv) \geq (iii)$ when $z(1) - z(2) \leq a_H c_1 - a_L c_1$ (case 1) but it becomes $(ii) \geq (i)$ and $(iii) \geq (iv)$ when $z(1) - z(2) \geq a_H c_1 - a_L c_1$ (case 2). This observation implies that there are two cases to be evaluated throughout the solution in the next section.

With these threshold cash levels, banks need to make up their minds whether to liquidate (bankrupt) or to continue in each state and determine their optimal cash levels, x_i. On the other hand, a social planner will determine the optimal network between banks by solving the following optimization problem:

$$\max_{z(0),z(1),z(2)} \sum_i \Pi_i(r(x_i^*, x_{3-i}^*))$$

$$\Pi_i(r(x_i^*, x_{3-i}^*))|_{[z(0),z(1),z(2)]} \geq \Pi_i(r(x_i^*, x_{3-i}^*))|_{[0,0,0]} \tag{3}$$

Here, the constraint in the optimization problem are the participation constraints of the banks that make sure that banks are better off with the network relationship compared to the case without it. The solution in the next section will compare the outcome with and without the network relationship and underline the efficiency resulting from this relationship.

3. Model Solution

3.1 Investment Portfolio Decisions

This section investigates the investment portfolio decisions of banks. As already briefly mentioned in the previous section, banks will evaluate their alternative payoffs in order to decide about their optimal cash levels.

Different levels of cash holdings and in return banks' liquidation versus continuation decisions in each state affect their payoffs via the return (r) of the project. In other words, r changes with respect to banks' strategies. Strategies are denoted by

$s_i^\omega \epsilon \{L, C\}$ for each bank $i \in \{1,2\}$ and each state $\omega \in \{1,2\}$, where L denotes liquidation at t_1 and C denotes continuation until the final date t_2. As outlined before, the liquidation return $\tilde{r}$ is independent from the other bank's liquidation versus continuation decision, meaning that $r_i^\omega(s_i^\omega = L, s_{3-i}^\omega \epsilon\{L, C\}) = \tilde{r}$. To repeat, the bank that does not hold enough cash for early consumers and so liquidates at t_1 declares bankruptcy since $\tilde{r} < 1$. However, if the bank does not liquidate, its return depends on the other bank's decision. That is to say, the banks that does not liquidate obtains $\underline{r}$ from the project at t_2 if the other bank liquidates in that state: $r_i^\omega(s_i^\omega = C, s_{3-i}^\omega = L) = \underline{r}$. Finally, the return of the non-liquidating bank jumps to $\overline{r}$ from the project at t_2 if the other bank also does not liquidate in that state: $r_i^\omega(s_i^\omega = C, s_{3-i}^\omega = C) = \overline{r}$.

Denote the payoff of bank *i* in state ω that realizes at t_2 as $\Pi_i^\omega(r_i^\omega(.,.))$. Accordingly, there are three alternative payoffs for bank i in state ω depending on the combined (liquidation/continuation) decisions of the two banks:

$$\Pi_i^\omega(r_i^\omega(s_i^\omega = L, s_{3-i}^\omega \hat{I}\{L,C\})) = \Pi_i^\omega(\tilde{r}) = 0 \tag{4}$$

$$\begin{aligned}\Pi_i^\omega(r_i^\omega(s_i^\omega = C, s_{3-i}^\omega = L)) &= \Pi_i^\omega(\underline{r}) \\ &= y_i \underline{r} + x_i \pm z(\omega) - EC_i(\omega) - LC_i(\omega)\end{aligned} \tag{5}$$

$$\begin{aligned}\Pi_i^\omega(r_i^\omega(s_i^\omega = C, s_{3-i}^\omega = C)) &= \Pi_i^\omega(\overline{r}) \\ &= y_i \overline{r} + x_i \pm z(\omega) - EC_i(\omega) - LC_i(\omega)\end{aligned} \tag{6}$$

Where $EC_i(\omega)$ and $LC_i(\omega)$ are payments made to early and late consumers, respectively: For state 1, $EC_1(\omega = 1) = a_H c_1$; $LC_1(\omega = 1) = (1 - a_H)c_2$ and , $EC_2(\omega = 1) = a_L c_1$, $LC_2(\omega = 1) = (1 - a_L)c_2$. Similarly, for state 2, $EC_1(\omega = 2) = a_L c_1$ and $LC_1(\omega = 2) = (1 - a_L)c_2$; $EC_2(\omega = 2) = a_H c_1$ and $LC_2(\omega = 2) = (1 - a_H)c_2$.

The expected payoff of bank i is denoted by $\Pi_i(\Pi_i^{\omega=1}(.), \Pi_i^{\omega=2}(.)) = p_1\Pi_i^{\omega=1}(.) + p_2\Pi_i^{\omega=2}(.)$. This expected payoff function is decreasing in the cash amount x_i since the coefficient of x_i is negative due to the fact that both of the possible return values $\underline{r}$ and $\overline{r}$ are bigger than one. This implies that the banks will optimally hold the minimum cash amount that is just enough to cover the early withdrawals in each state to prevent the early liquidation with zero payoffs. In other words, the banks will always keep the maximum of the threshold cash levels of the two possible cases that are introduced above. This intuition leads us to the following proposition.

Proposition 1. *Banks will invest in cash amount of* $x_1 = \max(a_H c_1 - z(1), a_L c_1 - z(2)$ and $x_2 = \max(a_L c_1 + z(1), a_H c_1 + z(2))$.

Proof. The expected payoffs of banks

$\Pi_i(\Pi_i^{\omega=1}(.), \Pi_i^{\omega=2}(.)) = p_1 \Pi_i^{\omega=1}(.) + p_2 \Pi_i^{\omega=2}(.)$ is decreasing in x_i. This could be easily understood if one inserts the value of $y_i = 1 - x_i \pm z(0)$ into the payoff function by using the budget equations of the banks. In this way, one realizes that the coefficient of x_i is either $1 - \underline{r}$ or $1 - \overline{r}$ depending on the realized payoff function that is determined by the continuation/liquidation decisions of banks as represented in equations (5) and (6). In both cases, the coefficient of x_i will be always negative in the payoff functions since $1 < \underline{r} < \overline{r}$.

That is the banks will not hold optimally more than the threshold levels that is just enough to prevent early liquidation in each state. If the banks hold the minimum of these two thresholds values, they would have to liquidate in one of the states. This is not optimal since $\min(\Pi_i^{\omega}(\overline{r}), \Pi_i^{\omega}(\underline{r})) > \Pi_i^{\omega}(\widetilde{r}) = 0$.

3.2 The Optimal Network Relationship (Financial Contract)

This section determines the optimal network relationship that occurs between banks. The aim is to determine first $\vec{z}$, which may be interpreted as either the payoff vector stemming from the 'committed credit lines' contract between the banks or the insurance payment of a bilateral insurance contract. Second, it is to determine the (net) upfront commitment fee of the contract or the net of insurance premia $z(0)$. While determining this contract, we will apply the social welfare approach. This means that we will consider the un-weighted sum of expected utilities of the two banks.

The previous section showed that the banks will hold the maximum of the threshold cash levels; i.e. $x_1 = \max(a_H c_1 - z(1), a_L c_1 - z(2)$ and $x_2 = \max(a_L c_1 + z(1), a_H c_1 + z(2))$. As it was introduced in the section of the model introduction, the magnitude of these two threshold levels depends on the payoff vector $\vec{z} = (z(1), z(2))$. Specifically, it depends on whether $z(1) - z(2) \le a_H c_1 - a_L c_1$ (case 1) or $z(1) - z(2) \ge a_H c_1 - a_L c_1$ (case 2). The next pages will study these two separate cases in order to determine the optimal contract between the banks.

3.2.1 Case 1

Under case 1, bank 1 and bank 2 hold $x_1 = a_H c_1 - z(1)$ and $x_2 = a_H c_1 + z(2)$ which are the sufficient cash amounts to cover the early withdrawals. Expected payoffs of banks with these specified cash holdings are denoted as follows:

$$\begin{aligned} & p_1 \Pi_i^{\omega=1}(r_i^1(s_i^1 = C, s_{3-i}^1 = C)) + p_2 \Pi_i^{\omega=2}(r_i^2(s_i^2 = C, s_{3-i}^2 = C)) \\ & = p_1 \Pi_i^{\omega=1}(\overline{r}) + p_2 \Pi_i^{\omega=2}(\overline{r}) \end{aligned} \tag{7}$$

Since neither of the banks liquidates in any of the states, they obtain the higher return $\overline{r}$ from the common project at t_2 in equation (6).

The sum of their expected payoffs,

$$p_1\left[\sum_i \Pi_i^{\omega=1}(\bar{r})\right]+p_2\left[\sum_i \Pi_i^{\omega=2}(\bar{r})\right] \tag{8}$$

at the optimal cash levels $x_1 = a_H c_1 - z(1)$ and $x_2 = a_H c_1 + z(2)$ is calculated by inserting the values of $\Pi_i^{\omega}(\bar{r})$ as represented in equation (6). This sum is equal to:

$$\begin{aligned}&(\bar{r}-1)\left[z(1)-z(2)\right]+2\bar{r}(1-a_H c_1)\\&+a_H c_1 - a_L c_1 - (2-a_H-a_L)c_2\end{aligned} \tag{9}$$

The sum of banks' expected payoffs in equation (9) is independent from the (net) upfront commitment fee, $z(0)$ and is strictly increasing in $z(1) - z(2)$ since $\bar{r} > 1$ under 'case 1' which occurs when $z(1) - z(2) \leq a_H c_1 - a_L c_1$.

3.2.2 Case 2

When it comes to 'case 2', bank 1 and bank 2 hold cash amounts of $x_1 = a_L c_1 - z(2)$ and $x_2 = a_L c_1 + z(1)$. In a similar way, one may calculate the sum of banks' expected payoffs at the optimal cash levels $x_1 = a_L c_1 - z(2)$ and $x_2 = a_L c_1 + z(1)$, which is equal to:

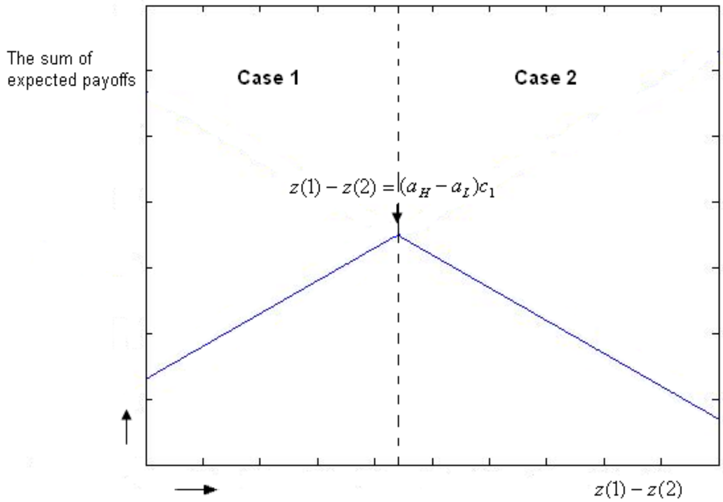
$$\begin{aligned}&(1-\bar{r})\left[z(1)-z(2)\right]+2\bar{r}(1-a_L c_1)\\&+a_L c_1 - a_H c_1 - (2-a_H-a_L)c_2\end{aligned} \tag{10}$$

The sum of banks' expected payoffs in equation (10) is again independent from $z(0)$ but is strictly decreasing in $z(1) - z(2)$ since $\bar{r} > 1$ under 'case 2' which occurs when $z(1) - z(2) \geq a_H c_1 - a_L c_1$.

To summarize, the sum of banks' expected payoffs (equation (9)) is strictly increasing in $(z(1) - z(2))$ when $z(1) - z(2) \leq a_H c_1 - a_L c_1$. and it is (equation (10)) strictly decreasing with the same slope when $z(1) - z(2) \geq a_H c_1 - a_L c_1$. And, it is clear that the sum of expected payoffs under 'case 1' and 'case 2' (equations (9) and (10)) are equal at the point $z(1) - z(2) = a_H c_1 - a_L c_1$. This is illustrated with Figure 2.

Accordingly, banks set optimally the difference of insurance payments $(z(1) - z(2))$ to the difference of early withdrawals between the two states that is $(a_H c_1 - a_L c_1)$ since the sum of their expected payoffs is increasing until this point but after that it starts to decrease. It means that 'case 1' and 'case 2' overlap at the optimum. In other words, if banks do not want to liquidate (go bankrupt) due to early withdrawals in any of the states and hold always enough cash for this purpose in equilibrium, they match the necessary cash amounts (threshold cash levels) of the two states with a liquidity network relationship. Banks zero out the variance of the liquidity shock due to early withdrawals by setting $z(1) - z(2) = a_H c_1 - a_L c_1$ with this relationship. In other words, they are able to get rid of the liquidity risk.

Figure 2 Sum of Expected Payoffs



Notes: This figure depicts how the sum of banks' expected payoffs changes with respect to the difference of net credit amounts exchanged by banks (or the difference of insurance payments in states 1 and 2). The payoff function strictly increases under 'Case 1' that occurs when z(1)-z(2) is smaller than ($a_Hc_1 - a_Lc_1$) and after that point it starts to decrease when z(1)-z(2) is bigger than ($a_Hc_1 - a_Lc_1$). Thus the optimal point for the total payoff function occurs at z(1)-z(2)= ($a_Hc_1 - a_Lc_1$) where 'Case 1' and 'Case 2' overlaps. In other words, banks match the necessary cash amounts of the two states to prevent early liquidation with the committed credit lines

At this point, it is interesting to point out that if the payoff vector $\vec{z}$ is set to be symmetric, meaning that $z(1)$ and $z(2)$ are equal in absolute terms, the solution becomes equivalent to the one in Allen and Gale (2000). Banks hold as cash an amount of $\frac{a_Hc_1+a_Lc_1}{2}$ and the bank that has a high demand for liquidity obtains an amount of $\frac{a_Hc_1-a_Lc_1}{2}$ from the other bank with the low liquidity shock. However, this is just a special case of the optimal network relationship in which the payoff vector $\vec{z}$ is symmetric. The next step is to determine the net of upfront commitment fee or the price of the contract, $z(0)$. This network relationship could take place only if both banks are willing to enter into it. In other words, they would prefer to enter into this relationship if they do at least as well as without this relationship. Thus, the (net) upfront commitment fee $z(0)$ should lie within a specific range in order to satisfy the participation constraints of both banks.

By employing these ideas, the next proposition presents the characteristics of the liquidity network relationship that takes place between the two banks.

Proposition 2. *It is optimal for banks to set* $z(1)-z(2)$ *to* $a_Hc_1 - a_Lc_1$ *and* $z(0)$ *lies in* $z(1)-\frac{(a_Hc_1-a_Lc_1)(\bar{r}-p_1)}{\bar{r}} \leq z(0) \leq z(1)-\frac{(a_Hc_1-a_Lc_1)p_2}{\bar{r}}$.

Proof. The condition regarding the payoff vector $\vec{z}$ is explained in the text. Here, the feasible range of $z(0)$ will be determined from the participation constraints of banks. In order that banks enter into a network relationship by which they dispose of the liquidity risk as described, they should not be worse off than they would be had they not constructed such a financial relationship. So, one should compare banks' equilibrium payoffs with the liquidity network relationship to the payoff they would have had if they had not settled such a relationship but again prevented early liquidation by holding a cash amount equal to $a_H c_1$. Accordingly, for bank 1 it should hold that its payoff with the network relationship,

$$p_1 \Pi_1^{\omega=1}\left(r_1^1(s_1^1 = C, s_2^1 = C)\right) + p_2 \Pi_1^{\omega=2}\left(r_1^2(s_1^2 = C, s_2^2 = C)\right) = p_1 \Pi_1^{\omega=1}(\bar{r}) + p_2 \Pi_1^{\omega=2}(\bar{r})$$

at the optimal cash level $x_1 = a_H c_1 - z(1) = a_L c_1 - z(2)$ is bigger or equal than the payoff it would have had if it were independent and held a cash amount of $x_1 = a_H c_1$. By writing down this idea in mathematical terms, we get the following:

$$(1 - z(0) - a_H c_1 + z(1))\,\bar{r} + a_H c_1 - z(1) + p_1 z(1) + p_2 z(2) - p_1 a_H c_1 - p_1 (1 - a_H) c_2 - p_2 a_L c_1 - p_2 (1 - a_L) c_2 \geq (1 - a_H c_1)\,\bar{r} + a_H c_1 - p_1 a_H c_1 - p_1 (1 - a_H) c_2 - p_2 a_L c_1 - p_2 (1 - a_L) c_2$$

This inequality yields $z(0) \leq z(1) - \frac{(a_H c_1 - a_L c_1) p_2}{\bar{r}}$ after substituting the equilibrium condition that is $z(1) - z(2) = a_H c_1 - a_L c_1$.

By following analogous steps, one may obtain the participation constraint of bank 2, which reads as follows:

$$(1 + z(0) - a_H c_1 - z(2))\,\bar{r} + a_H c_1 + z(2) - p_1 z(1) - p_2 z(2) - p_1 a_L c_1 - p_1 (1 - a_L) c_2 - p_2 a_H c_1 - p_2 (1 - a_H) c_2 \geq (1 - a_H c_1)\,\bar{r} + a_H c_1 - p_1 a_L c_1 - p_1 (1 - a_L) c_2 - p_2 a_H c_1 - p_2 (1 - a_H) c_2$$

From this inequality, one obtains $z(0) \geq z(1) - \frac{(a_H c_1 - a_L c_1)(\bar{r} - p_1)}{\bar{r}}$.

Proposition 2 shows that the (net) upfront commitment fee or the price of the contract is dependent on idiosyncratic risks of the banks. The range which includes the feasible values of $z(0)$ shifts rightward as bank 1 gets riskier, i.e. the probability that bank 1 is hit by a high liquidity shock (p_1) gets bigger. Put differently, the upfront commitment fee $z(0)$ increases as bank 1 (2) gets riskier (more secure) meaning that p_1 increases (p_2 decreases). To get the intuition behind this result, remember that $z(0)$ is the price of the 'committed credit line' contract that bank 1 pays to bank 2 as the buyer of the contract. Hence, bank 1 faces a higher price if it is riskier. From the viewpoint of bank 2, it is compensated by collecting a higher fee if it is less risky. This result is in accordance with relatively new literature, which shows that risk-based pricing takes place in the interbank market. Compared to the more traditional view, which presumes there is a single rate in the interbank market for all borrowers, this strand of literature has been able to empirically demonstrate that high-risk banks have consistently paid more than safer banks. (King, 2008, Furfine, 2001).

To conclude, this section has shown that banks dispose of their liquidity risk by equating the necessary cash amounts for the two states. Moreover, the two banks need to hold in total less cash for early consumers compared to the situation where they do not establish such a network but stay independent. More precisely, with this network the two banks hold in total $\sum_i x_i = 2a_H c_1 + z(2) - z(1) = 2a_L c_1 + z(1) - z(2) = a_H c_1 + a_L c_1$. However, the two banks in total would need to hold $2a_H c_1$ for early consumers if they had stayed independent. That is to say, the total required cash holdings of the whole banking system decreases by means of this network relationship.

4. The Case of Imperfectly Negatively Correlated Liquidity Shocks

Previous sections focused on the liquidity network relationship that the two banks establish when they face perfectly negatively correlated liquidity shocks. It is shown that banks eliminate their liquidity risk by equating the necessary cash amounts to prevent an early liquidation for each state with such a financial relationship. However, it would not be possible for banks to reduce the variance of the liquidity shock to zero if they faced imperfectly correlated liquidity shocks. Then, it is interesting to ask whether banks would prefer to enter into a similar network relationship even with imperfectly negatively correlated liquidity shocks under which they cannot eliminate the liquidity risk.

Accordingly, consider again two banks in two separate regions as before. All of the model specifications remain the same, except that there exists a third state where both banks face a high proportion of early consumers. Then, the proportions of early consumers with respect to two regions and three states are as follows:

$$\text{The proportion of early consumers} =: \left\{ \begin{array}{l} a_H \text{ in } R_1, a_L \text{ in } R_2 \text{ with } p_1 \\ a_L \text{ in } R_1, a_H \text{ in } R_2 \text{ with } p_2 \\ a_H \text{ in } R_1, a_H \text{ in } R_2 \text{ with } p_3 \end{array} \right\} \quad (11)$$

where $\sum_i p_i = 1$.

It is clear that banks cannot be insurers of each other in the third state since they both realize a high liquidity shock. Thus, they are never able to completely get rid of the liquidity risk as they are with the perfectly negatively correlated two states.

The result of the previous section shows us that banks will prefer to match the threshold cash levels of the states $\omega = 1$ and $\omega = 2$ that are required to prevent an early liquidation (bankruptcy) if they enter into a network relationship. In this respect, banks' liquidation versus continuation decisions with imperfectly negatively correlated three states are as follows: I) the bank liquidates in all three states if it prefers to hold a cash amount that is less than the necessary amount to be able to cover the early withdrawals of the states $\omega = 1$ and $\omega = 2$, II) the bank liquidates only in the third state $\omega = 3$ if it prefers to hold a cash amount that is enough to cover the early withdrawals of the states $\omega = 1$ and $\omega = 2$ but not sufficient to cover the early withdrawals in state $\omega = 3$ that is $a_H c_1$ and III) the bank does not liquidate in any of the states if it prefers to hold a cash amount of $a_H c_1$ that is enough to cover early withdrawals in any state.

By following the same logic presented in proposition 1 that is $\min\left[\Pi_i^\omega(\bar{r}), \Pi_i^\omega(\underline{r})\right] > \Pi_i^\omega(\tilde{r}) = 0$, the banks will hold $x_1 = \max[a_H c_1 - z(1) = a_L c_1 - z(2), a_H c_1] = a_H c_1$ and $x_2 = \max[a_L c_1 + z(1) = a_H c_1 + z(2), a_H c_1] = a_H c_1$. In other words, banks will set their cash amounts optimally to $x_1 = x_2 = a_H c_1$ under the case of imperfectly negatively correlated shocks.

In this situation, realize that if banks hold optimally the maximum of the necessary amounts to prevent an early liquidation in each state, i.e. $x_i = a_H c_1$ as cash, the liquidity network relationship does not provide any gain from the social efficiency point of view, considering the fact that there is no reduction in the total cash amount required to prevent early liquidation. The banks could have stayed independent and they would have still prevented early liquidation by holding the same amount of cash i.e., $x_i = a_H c_1$. While the liquidity network relationship is beneficial and induces a reduction in the total required cash holdings of the whole banking system when banks are confronted with perfectly negatively correlated liquidity shocks, this is not the case with imperfectly negatively correlated liquidity shocks. However, the next proposition shows that there still exists a certain value of the (net) upfront commitment fee (price) $z(0)$ that makes banks indifferent between staying independent and forming a network relationship. That means even if there is no efficiency gain of the network relationship, the banks may still form one that leaves them indifferent. This is an interesting result, giving an idea why the modern financial system is too much inter-connected, which may cause contagion effects in the occurrence of sudden shocks. Basically, the interconnection between banks is even robust to the extreme case in which there is no direct gain from such a financial relationship. Despite the fact that there is still a price for the committed credit lines contract that the banks can mutually agree on, the set of feasible prices shrinks when the shocks of banks get less negatively correlated. Put differently, the interbank market gets narrowed when banks experience common shocks like in a crisis situation. This is similar to what has been observed during the financial crisis of 2007-2009, when the interbank market lived through a freeze due to the common 'precautionary' demand for liquidity by banks. (Acharya and Skeie, 2011)

As already discussed above, whether banks with optimal cash amounts of $x_i = a_H c_1$ will settle a network relationship in which they equate the threshold values that are required to continue the project in t_2 for states $\omega = 1$ and $\omega = 2$, depends on the value of the upfront commitment fee, $z(0)$. Proposition 3 shows that the interval that happens to contain the feasible values of $z(0)$, which make the banks prefer the network relationship over staying independent, shrinks to a single value under the case of imperfectly negatively correlated liquidity shocks. Thus, the banks set the value of the upfront commitment fee $z(0)$ to this certain value which is determined below. This network relationship makes them as good as staying independent. However, existence of a single value for the upfront commitment fee should not create a concern whether it might be an unstable solution. Since the banks are indifferent between staying independent and forming a network, it would imply that each outcome is realized fifty percent of the time.

Proposition 3. *The (net) upfront commitment fee $z(0)$ takes a single value, i.e. $z(0) = \frac{(1-p_3)z(1)-(a_H c_1 - a_L c_1)p_2}{\bar{r}}$ for the existence of a liquidity network relationship between banks.*

Proof. The feasible value of $z(0)$ is determined from the participation constraints of banks: They should be satisfied in order to have a network relationship between banks in which the threshold cash levels of the states $\omega = 1$ and $\omega = 2$ that are required to prevent an early liquidation are equal, i.e.
$z(1) - z(2) = a_H c_1 - a_L c_1$ · In other words, banks should not be worse off than they would be had they not constructed such a financial relationship. Thus, the task is to compare banks' equilibrium payoffs with the network relationship to the payoff they

would have had if they stayed independent but again prevented early liquidation with a cash amount of $x_i = a_H c_1$.

Accordingly, for bank 1 it should hold that its payoff with the network relationship,

$$\sum_{\omega} p_\omega \Pi_1^\omega \left(r_1^\omega (s_1^\omega = C, s_2^\omega = C)\right)$$

at the optimal cash level $x_1 = a_H c_1$ summed over the two states is bigger than or equal to than the payoff it would have had if it stayed independent and held again a cash amount of $x_1 = a_H c_1$:

$(1 - z(0) - a_H c_1)\,\overline{r} + a_H c_1 + p_1 z(1) + p_2 z(2) - (1-p_2) a_H c_1 - (1-p_2)(1-a_H) c_2 - p_2 a_L c_1 - p_2 (1-a_L) c_2 \geq (1 - a_H c_1)\,\overline{r} + a_H c_1 - (1-p_2) a_H c_1 - (1-p_2)(1-a_H) c_2 - p_2 a_L c_1 - p_2 (1-a_L) c_2$

This inequality yields $z(0) \leq \frac{(1-p_3)z(1)-(a_H c_1 - a_L c_1)p_2}{\overline{r}}$ when the equilibrium condition, $z(1) - z(2) = a_H c_1 - a_L c_1$ is substituted.

Similarly, the participation constraint of bank 2 is presented with the next inequality:

$(1 + z(0) - a_H c_1)\,\overline{r} + a_H c_1 - p_1 z(1) - p_2 z(2) - p_1 a_L c_1 - p_1 (1-a_L) c_2 - (1-p_1) a_H c_1 - (1-p_1)(1-a_H) c_2 \geq (1 - a_H c_1)\,\overline{r} + a_H c_1 - p_1 a_L c_1 - p_1 (1-a_L) c_2 - (1-p_1) a_H c_1 - (1-p_1)(1-a_H) c_2$

From this inequality, one obtains $z(0) \geq \frac{(1-p_3)z(1)-(a_H c_1 - a_L c_1)p_2}{\overline{r}}$ by substituting the equilibrium condition, $z(1) - z(2) = a_H c_1 - a_L c_1$.

5. Conclusion

Banks or financial institutions that operate in separate regions and thus are likely to face different liquidity shocks may enter into a network relationship. Different than the previous literature, this paper perceives this network relationship as 'committed credit lines' contract in the market created by banks without any additional restrictions. Then, the price and the payoff vector of this contract, which perfectly insures banks against liquidity shocks, is determined optimally in equilibrium. It is seen that the price or the upfront commitment fee depends on the idiosyncratic shocks of the banks, which is consistent with some recent empirical evidence of a risk-based pricing in the interbank market. Banks are also able to decrease the total required cash holdings of the banking system for the liquidity needs of the early consumers.

Subsequently, the paper looks at the case where banks have imperfectly negatively correlated liquidity shocks. It is not possible anymore for banks to zero out the variance of the liquidity shock. Despite the fact that banks cannot eliminate the liquidity risk completely, they may still establish a liquidity network relationship with which banks insure each other in hedgeable states. This is true even when there is no direct efficiency gain from this relationship in terms of the total required cash holdings to prevent early liquidation. The reason for that is that there still exists an agreeable price for the contract, which leaves both banks indifferent between staying independent and entering into this relationship. But this also means that the set of feasible prices shrinks to a single value when banks face common shocks.

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