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Pension Demand and Utility: The Life Annuity Puzzle*

Tomáš CIPRA—Faculty of Mathematics and Physics, Charles University of Prague
(cipra@karlin.mff.cuni.cz)

Abstract
This paper shows by means of the concept of utility that annuitization through life annuities or a pension can be an efficient instrument for the economic assurance of seniors. Various quantitative arguments are presented supporting this statement (e.g. annuity equivalent wealth [AEW] is calculated using Czech data). In conclusion, some practical arguments are mentioned regarding why the real demand on life annuities contradicts this result so that a so-called annuity puzzle arises in pension practice.

1. Introduction

The quantitative analysis of pensions can be either actuarial or economic. This paper investigates pensions and life annuities more from the point of view of economists, i.e. it deals with the economic theory of pensions (see, for example, Sheshinski, 2008) even if the contribution brings various quantitative results (for the actuarial approach to pensions see, for example, Booth et al., 1999; Cipra, 2012; Koller, 2000; Lee, 1986; McGill, 1975; and Winklevoss, 1977). In particular, this paper deals with the economic theory of pension demand and pension utility and looks for optimal consumption strategies for individuals in old age. The main message of this paper is to show that annuitization (i.e. a consumption strategy in the form of life annuities) is a powerful instrument for the economic assurance of seniors, and this text sets forth numerous quantitative arguments to back up such a conclusion. On the other hand, since the demand for annuity products in practice is not compatible with this result, the paper strives to deliver practical arguments explaining this "annuity puzzle".

The structure of this paper is as follows: Yaari's approach to pension demand is presented in Section 2. This theory is modified to be compatible with the utility approach to pensions and annuity markets in Section 3. Here the basic utility functions including various ways of discounting consumption strategies including annuity instruments are defined. Section 4 looks for optimal consumption strategies under consumption restrictions given as possibilities to enter bond or annuity markets. In addition, the analytical formulas for annuity equivalent wealth AEW_0 are derived for bond and annuity markets. Some numerical results using data for the Czech Republic are presented in Section 5. Finally the concluding Section 6 strives to deliver some arguments resolving the annuity puzzle.

2. Pension Demand

The model approach to pension demand was initiated by Yaari (1965), who for the first time took into account in the framework of pension-demand analysis

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the randomness of life expectancy, i.e. the randomness of the pension decumulation period. He came to a basic result that under standard assumptions maximum annuitization is the best scenario so that one should prefer pension products to immediate consumption. Obviously, that does not correspond to actual practice, where the demand for annuity products is usually surprisingly weak (at least if they are not backed up by massive state support or if they are not even social benefits). Why this is not the case is an anomalous annuity puzzle for economists (see Section 6). Since Yaari's approach is important for this text, we describe it graphically under simplifying assumptions (it is not difficult to extend it so that it is closer to reality). The simplifying assumptions for a given individual in the framework of Yaari's approach are as follows:

(A1) The only randomness consists in the random length of a human life. The random character of other factors (e.g. interest rates or inflation rates) is not taken into account (on the other hand, the randomness of inflation, for example, can be eliminated by using indexed annuities).

(A2) Pension payments are paid out at most at two time points denoted for simplicity as 0 and 1: at time 0 (usually the date of reaching the pension age, say 65) the individual is alive with probability 1, while the event of survival until time 1 is not certain (with probability $p < 1$).

(A3) The individual keeps at his or her disposal capital W_0 at time 0 without any supplementary capital sources.

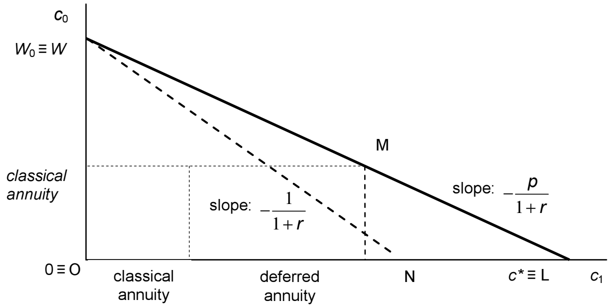
(A4) The investment interest rate between times 0 and 1 is fixed at r. One cannot invest in risky assets (e.g. stocks) to convey capital to inheritors and to borrow money using the future annuity payment as collateral.

(A5) Capital W_0 can be used in several ways that can complete each other: (*i*) in periods 0 and 1 one uses amounts c_0 and c_1 for direct consumption (see, for example, *Figure 1*); (*ii*) a part of W_0 can be invested in simple saving products denoted for simplicity as a bond (this may also be bank deposits or savings accounts under the condition that they are not influenced by the mortality behavior of individuals; (*iii*) a part of W_0 can be used to purchase a life annuity that pays out constant pension payments of A per each unit of the capital W_0 at time 0 (i.e. immediately if the annuity is not deferred) and at time 1 (i.e. if the individual survives until time 1, which occurs according to (A2) with the probability p); if the annuity is deferred, then only one payment A^{def} occurs at time 1.

Figure 1 plots the possible consumption c_0 at time 0 (the horizontal axis) and the possible consumption c_1 at time 1 (the vertical axis) under the assumptions (A1)-(A5). Moreover, it gives the consumption restrictions for the given individual, meaning that any consumption in the northeasterly direction is infeasible:

(1) The triangle ONW bounds the region of possible consumptions using only bonds (i.e. saving instruments) without annuities. For instance, if one consumes all of capital W_0 at time 0 (i.e. $c_0 = W_0$), then nothing remains to consume at time 1 (i.e. $c_1 = 0$), and this situation corresponds to the vertex W of the region ONW. Similarly, if one consumes nothing at time 0 (i.e. $c_0 = 0$) then the maximum consumption at time 1 can amount to as much as $W_0 \cdot (1 + r)$, and this situation corresponds to the vertex N of the region ONW.

Figure 1 Graphical Plot of Consumptions c_0 and c_1 at Times 0 and 1 Describing the "Annuity Puzzle"



(2) However, the region ONW is only a subset of the maximum consumption region OLW, which is attainable by means of annuities deferred to time 1. The annuity payment A^{def} per unit of initial capital W_0 (i.e. under the condition of surviving until time 1) must fulfill $p \cdot A^{def}/(1+r) = 1$ so that

$$A^{def} = \frac{1+r}{p} \tag{2.1}$$

This implies that the vertex L in *Figure 1* of the region OLW corresponds to the maximum consumption

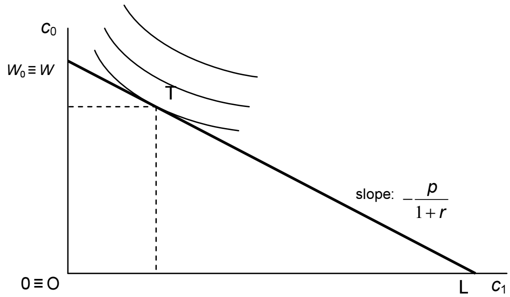
$$c^* = \frac{1+r}{p} \cdot W_0 \tag{2.2}$$

at time 1 (in this situation all of the accessible capital W_0 at time 0 is used to purchase the deferred annuity; see also the slope in *Figure 1*). Here one can see the difference from saving investments in bonds or deposits since the investment effect in the framework of annuities is supported by means of so-called mortality drag following from inequality $p < 1$. The mortality drag guarantees higher investment yields but only for those who survive until the end of investment period, which is a trivial conclusion.

(3) It is also obvious from *Figure 1* that classical (i.e. "non-deferred") annuities do not suffice for the maximum consumption result.

The previous analysis does not respect two aspects which are important for pension demand (moreover, the annuity puzzle is discussed from the practical point of view in Section 6). The first of these aspects consists in the assumption that a perfect annuity market exists, which is a non-productive ideal in practice. The second one is the fact that only such points in *Figure 1,* which are optimal in the sense of the north-easterly direction, should be preferred, i.e. the analysis has so far ignored individual preferences. In particular, individuals can prefer future consumption strategies according to their indifference curves. These curves are an important instrument of the utility theory (see Section 3): each individual has a specific system of indifference curves which can be defined by means of a suitable utility function in the sense that all points of the same indifference curve are equally acceptable for the given individual. A reasonable system of indifference curves is plotted in *Figure 2*, where the tangen-

Figure 2 Graphical Plot of Consumptions *c*0 and *c*1 at Times 0 and 1 Including Point T of Optimal Consumption of a Given Individual



tial point T in the sense of the northeasterly direction is the point of optimal consumption for the given individual (it respects both the consumption preferences and the consumption constraints).

If one wants to generalize the previous result, one should distinguish three types of markets (this will also be important for the utility pension theory in Section 3, where these markets appear in various combinations):

- *perfect bond market PB* enables investing immediately any amount in bonds with arbitrary maturity;
- *perfect annuity market PA* enables investing immediately any amount in life annuities with arbitrary deferment (e.g. the situations in *Figures 1* and *2* are possible only under the *PA* assumption);
- *conventional annuity market CA* enables investing immediately any amount in classical annuities (with fixed annuity payments).

3. Pension Utility

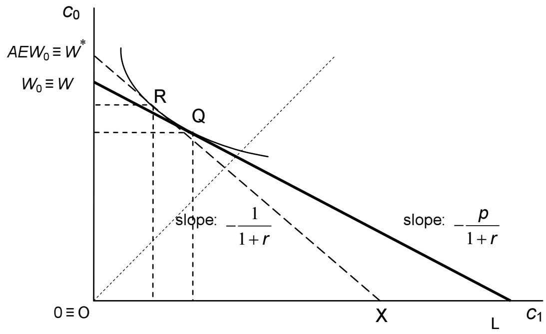
In Section 2 we stressed that the pension utility should be quantified separately for particular individuals. Since individual risk aversion commonly plays an important role in the consumption models involving life annuities, one should apply in the framework of pension utilities mainly such utility functions that involve this risk aspect (investigations of this type were initiated by Arrow, 1965, and Pratt, 1964).

The idea of utility functions is simple: a utility function $u(c)$ is a function transforming value c (i.e. an individual's wealth capitalized in his or her assets) into opportune consumption benefits for the individual; one can look upon this as an individual's utility rating. For instance, one of the most popular utility function is the logarithmic function $u(c) = \ln(c)$ $(c > 0)$, as it is increasing and concave so that it can describe the marginal utility effect (the utility of assets increases with their volume but by a decreasing rate, e.g. an allocation of flat social benefits does not represent for a prosperous individual with high income the same utility accrual as for a needy individual with low income). Generally, a utility function $u(c)$ should fulfill

$$u'(c) > 0, u''(c) < 0, u'''(c) > 0 \tag{3.1}$$

($c > 0$, see Arrow, 1965, and Pratt, 1964).

Figure 3 Graphical Construction of Annuity Equivalent Wealth AEW_0



In the pension context, one frequently applies the utility function

$$u(c) = \begin{cases} \dfrac{c^{1-\gamma}}{1-\gamma} & \text{for } \gamma > 0,\ \gamma \neq 1 \\ \ln c & \text{for } \gamma = 1 \end{cases} \tag{3.2}$$

($c > 0$) which involves the logarithmic utility function for $\gamma \to 1$ (γ is a fixed or estimated parameter). The utility function (3.2) is denoted as CRRA (*constant relative risk aversion*) since its *relative risk aversion* (or sometimes also *Arrow-Pratt measure*) $RRA(c)$ is constant

$$RRA(c) = -\frac{c \cdot u''(c)}{u'(c)} = \gamma \tag{3.3}$$

It is well known (see, for example, Arrow, 1965) that the utility function CRRA in (3.2) has all properties (3.1), and therefore we take advantage of it in this text.

The possibility of annuitization (i.e. purchasing a pension with suitable technical parameters) extends the consumption alternatives of individuals by allocating their assets from bonds to life annuities (see Section 2). The combination of these results with the utility approach is possible through the concept of annuity equivalent wealth, as described below.

Let an individual possessing wealth W_0 enter the annuity markets of the PA or CA type (see Section 2). Then annuity equivalent wealth AEW_0 is such a capital volume that is is necessary in order that the individual's utility will remain the same when only access to bond markets of the PB type is possible (i.e. when only saving activities are possible without any annuity instruments). It is obvious (according to Yaari's result from Section 2) that $AEW_0 > W_0$. This inequality can also be explained by means of *Figure 3*. Here the annuity equivalent wealth is constructed graphically in the following way: first, one selected the indifference curve tangential to the hypotenuse WL with the slope $-p/(1+r)$, which represents the consumption boundary in the case of perfect annuity market PA (the corresponding tangential point is denoted as Q). Then the tangent W^*X with the slope $-1/(1+r)$ was constructed for this selected indifference curve representing the consumption boundary in the case of perfect bond market PB (the corresponding tangential point is denoted as R).

The point W^*, which is constructed as the intersection of this tangent with the vertical axis, obviously corresponds to annuity equivalent wealth AEW_0.

Moreover, *Figure 3* enables investigation of the impact of survival probability p on the pension utility. If this probability p is low, then the hypotenuse WL is rather flat and differs significantly from the steep slope of tangent W^*X; this implies the positive role of annuitization, since annuity equivalent wealth AEW_0 is significantly higher than initial capital W_0 (in other words, the necessary capital which guarantees the same utility indifference curve with access only to PB markets is much higher than in the case of annuitization). Conversely, if this probability p is high, then the hypotenuse WL has a slope similar to the tangent W^*X; this implies that the annuitization benefit is not pronounced.

4. Optimization of Pension Utility Functions

So far we have ignored the fact that a typical feature of pensions is long time horizons (commonly several decades). This brings further aspects to the previous analysis, e.g. whether the given individual prefers immediate consumption to deferred consumption.

In any case, we must extend the previous consumption model with the two time points 0 and 1 only in the following way. The model will evaluate for an individual at time 0 (this time point usually corresponds to the pension age, e.g. 65) the individual's utility following from future flows of consumption services $c_0, c_1, \ldots, c_T$ which are guaranteed through a suitable investment of capital W_0 at time 0 in bonds (including classical saving activities) and in life annuities (i.e. in PB and PA markets defined in Section 2). Here consumption c_t at time t occurs only under the condition of surviving until this time so that, in particular, the number T of future consumptions is random (T is a random value denoted as remaining life expectancy respecting survival probabilities from the given Life Tables). If decisions of the individual are based on the individual's utility function $u(\cdot)$, then one must not only evaluate utility $u(c_t)$ of consumption c_t for each time t but, moreover, the final decision should respect the fact that the consumption services are ordered in time. In general, the corresponding decision task can be formulated as an optimization problem where we maximize (over future consumption flows $c_0, c_1, \ldots, c_T$ feasible from initial capital W_0) the expected value in the form of the objective function

$$v\left(c_0, c_1, \ldots, c_T\right) = \mathrm{E}_0\left\{u\left(c_0, c_1, \ldots, c_T\right)\right\} \tag{4.1}$$

where $u(c_0, c_1, \ldots, c_T)$ is a suitable joint utility of consumption strategies $c_0, c_1, \ldots, c_T$ and the symbol E_0 emphasizes the fact that the expected value is based on the information known at time 0 (see also Cannon and Tonks, 2008).

For practical calculations the objective function (4.1) must be specified. One usually preserves the principle of time additive separability TAS where the aggregate utility can be composed of the utilities in particular future times. In the simplest case we obtain the objective function denoted as v^{GD} (geometric discounting) in the form

$$v^{GD}\left(c_0, c_1, \ldots, c_T\right) = \sum_{i=0}^{T} \delta^i \cdot s_i \cdot u\left(c_i\right) \tag{4.2}$$

where δ is a constant between zero and one, s_i is a simplified notation for the survival probability from time 0 to time i (e.g. if time 0 corresponds to the pension age of 65, then a more sophisticated denotation for s_i is ${}_ip_{65}$) and $u(\cdot)$ is a utility function (of a single variable) of the individual (e.g. the utility function CRRA according to (3.2)). Then it holds:

(i) v^{GD} respects the principle of time additive separability TAS (see above);

(ii) v^{GD} respects the time preferences by means of the geometric discounting δ^i, where the constant δ represents the "degree of impatience" of the given individual (this can be looked upon as a subjective discounting factor that enables comparison of utilities of future consumption just by beginning at time 0: an impatient individual preferring to consume as early as possible has a coefficient δ lower than a more patient individual whose consumption can be deferred or distributed more to later years of the possible time horizon;

(iii) v^{GD} has the expected value in (4.1) in the form of a weighted mean with survival probabilities s_i as weights.

The objective function v^{GD} defined by (4.2) can be modified in various ways. Let's give two possible modifications. The first of them is the objective function denoted as v^{QHD} (quasi-hyperbolic discounting)

$$v^{QHD}\left(c_0, c_1, ..., c_T\right) = u\left(c_0\right) + \sum_{i=1}^{T} \beta \cdot \delta^i \cdot s_i \cdot u\left(c_i\right) \tag{4.3}$$

having an additional positive constant β. In this case the discounting ratio between neighboring summands ceases to be constant since it is $\beta \cdot \delta / 1 = \beta \cdot \delta$ between times 0 and 1 while it is further $\beta \cdot \delta^{i+1} / \beta \cdot \delta^i = \delta$ between times i and $i + 1$ for $i > 0$.

The second modification is the objective function denoted as v^{HD} (hyperbolic discounting)

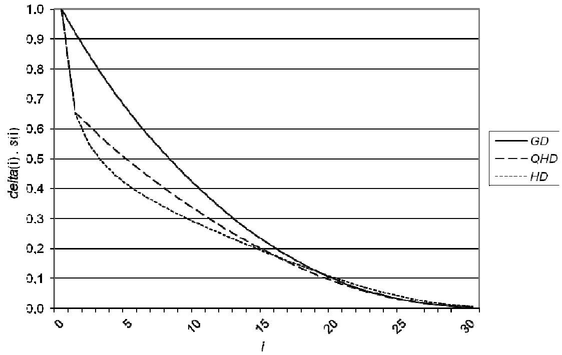
$$v^{HD}\left(c_0, c_1, ..., c_T\right) = \sum_{i=0}^{T} \frac{1}{(1+\eta \cdot i)^{\xi/\eta}} \cdot s_i \cdot u\left(c_i\right) \tag{4.4}$$

The hyperbolic decrease of the discounting sequence to zero modeled in (4.4) by means of two positive constants η and ξ is slower than in (4.2) and (4.3).

The corresponding discounting sequences (denoted in the objective functions of the types v^{GD}, v^{QHD} and v^{HD} generally as $\{\delta_i\}$) have the following form (in *Figure 4* they are weighted by survival probabilities as $\{\delta_i \cdot s_i\}$):

$$\begin{aligned} &\text{GD:} && \left\{\delta^i\right\},\ i = 0, 1, ..., T \\ &\text{QHD:} && \begin{cases} 1,\ i = 0 \\ \left\{\beta \cdot \delta^i\right\}, \quad i = 1, ..., T \end{cases} \\ &\text{HD:} && \frac{1}{(1+\eta \cdot i)^{\xi/\eta}},\ i = 0, 1, ..., T \end{aligned} \tag{4.5}$$

Figure 4 Probability Weighted Discounting Sequences {$\delta_i \cdot s_i$} for Objective Functions of the Types GD, QHD and HD
(GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1; Life Tables for Males in the Czech Republic in 2010)



The constants have been chosen according to the recommendation from Angeletos et al. (2001) and the survival probabilities s_i are taken from the male Life Tables of the Czech Republic in 2010.

Now one should solve the optimization problem maximizing the objective function of general form

$$\sum_{i=0}^{T} \delta_i \cdot s_i \cdot u(c_i) \tag{4.6}$$

(in particular, the sequence {δ_i} can be one of the sequences given in (4.5)) over feasible strategies of future flows of consumption services $c_0, c_1, \ldots, c_T$, which are guaranteed through a suitable investment of initial capital W_0 at time 0. The feasibility of future consumption services is given by consumption constraints which depend not only on the market type (PB, PA or CA, see Section 2) but also on investment yields (interest rates) in these markets. For simplicity, let's denote the annual yield in year i as r_i so that the amount due at the end of year i per unit initial capital is

$$R_i = \prod_{j=1}^{i} \left(1 + r_j\right)^j, \quad R_0 = 1 \tag{4.7}$$

The described decision task will be solved separately for PB, PA and CA markets but only for the utility function CRRA, which is frequent in practice (see (3.2)). Hence the objective function (4.1) to be maximized has the form

$$\sum_{i=0}^{T} \delta_i \cdot s_i \cdot \frac{c_i^{1-\gamma}}{1-\gamma} \tag{4.8}$$

(if $\gamma \neq 1$, while for $\gamma = 1$ one must replace the fraction with $\ln(c_i)$ in (4.8)).

4.1 Optimal Consumption in the Case of Perfect Bond Market PB

In this case, the optimal consumption strategies $c_0, c_1, \ldots, c_T$ can be obtained by maximizing the objective function (4.8) over the consumption constraints since

$$\sum_{i=0}^{T} c_i \cdot R_i^{-1} = W_0 \tag{4.9}$$

the only feasible investments of initial capital W_0 are savings in this PB case. The corresponding Lagrange function has the following form

$$L = \sum_{i=0}^{T} \delta_i \cdot s_i \cdot \frac{c_i^{1-\gamma}}{1-\gamma} + \lambda \cdot \left(W_0 - \sum_{i=0}^{T} c_i \cdot R_i^{-1} \right) \tag{4.10}$$

(λ is the corresponding Lagrange multiplier) with the following equations for zero partial derivatives

$$\frac{\partial L}{\partial c_i} = \delta_i \cdot s_i \cdot c_i^{-\gamma} - \lambda \cdot R_i^{-1} = 0, \;\; i = 0, 1, ..., T \tag{4.11}$$

Since $\delta_0 = s_0 = R_0 = 1$, one obtains $\lambda = c_0^{-\gamma}$ so that

$$c_i = c_0 \left(\delta_i \cdot s_i \cdot R_i \right)^{1/\gamma}, \quad i = 0, 1, ..., T \tag{4.12}$$

Hence substituting to (4.9) it follows that

$$c_0 = \frac{W_0}{\sum_{j=0}^{T} \left(\delta_j \cdot s_j \cdot R_j^{1-\gamma} \right)^{1/\gamma}}$$

so that by substituting to (4.12) the optimal consumption strategies are

$$c_i = \frac{W_0 \cdot \left(\delta_i \cdot s_i \cdot R_i \right)^{1/\gamma}}{\sum_{j=0}^{T} \left(\delta_j \cdot s_j \cdot R_j^{1-\gamma} \right)^{1/\gamma}}, \;\; i = 0, 1, ..., T \tag{4.13}$$

providing the following maximum value of the objective function

$$\max \left\{ \sum_{i=0}^{T} \delta_i \cdot s_i \cdot \frac{c_i^{1-\gamma}}{1-\gamma} \right\} = \frac{W_0^{1-\gamma}}{1-\gamma} \cdot \left(\sum_{i=0}^{T} \delta_i^{1/\gamma} \cdot s_i^{1/\gamma} \cdot R_i^{(1-\gamma)/\gamma} \right)^{\gamma} = \frac{W_0^{1-\gamma}}{1-\gamma} \cdot \Phi^{PB} \tag{4.14}$$

where

$$\Phi^{PB} = \left(\sum_{i=0}^{T} \delta_i^{1/\gamma} \cdot s_i^{1/\gamma} \cdot R_i^{(1-\gamma)/\gamma} \right)^{\gamma} \tag{4.15}$$

In the case of the logarithmic utility function with $\gamma = 1$ in (4.8) and with the objective function

$$\sum_{i=0}^{T} \delta_i \cdot s_i \cdot \ln \left(c_i \right) \tag{4.16}$$

the optimal consumption strategies are

$$c_i = \frac{W_0 \cdot \delta_i \cdot s_i \cdot R_i}{\sum_{j=0}^{T} \delta_j \cdot s_j}, \quad i = 0, 1, ..., T \tag{4.17}$$

providing the following maximum value of the objective function

$$\max \left\{ \sum_{i=0}^{T} \delta_i \cdot s_i \cdot \ln(c_i) \right\} = \sum_{i=0}^{T} \delta_i \cdot s_i \cdot \ln \left(\frac{W_0 \cdot \delta_i \cdot s_i \cdot R_i}{\sum_{j=0}^{T} \delta_j \cdot s_j} \right) \tag{4.18}$$

(Some formulas in Section 4 are not brand new, but all of them have been checked in this paper to eliminate errors if one accepts them without a qualified derivation.)

4.2 Optimal Consumption in the Case of Perfect Annuity Market PA

In this case the optimal consumption strategies $c_0, c_1, \ldots, c_T$ must fulfill the following consumption constraints

$$\sum_{i=0}^{T} s_i \cdot c_i \cdot R_i^{-1} = W_0 \tag{4.19}$$

since the investment of initial capital W_0 in life annuities must respect the probabilities of surviving between times 0 and i. Analogously as for PB markets, one obtains the optimal consumption strategies

$$c_i = \frac{W_0 \cdot (\delta_i \cdot R_i)^{1/\gamma}}{\sum_{j=0}^{T} s_j \cdot \left(\delta_j \cdot R_j^{1-\gamma}\right)^{1/\gamma}}, \quad i = 0, 1, ..., T \tag{4.20}$$

providing the following maximum value of the objective function

$$\max \left\{ \sum_{i=0}^{T} \delta_i \cdot s_i \cdot \frac{c_i^{1-\gamma}}{1-\gamma} \right\} = \frac{W_0^{1-\gamma}}{1-\gamma} \cdot \Phi^{PA} \tag{4.21}$$

where

$$\Phi^{PA} = \left(\sum_{i=0}^{T} \delta_i^{1/\gamma} \cdot s_i \cdot R_i^{(1-\gamma)/\gamma} \right)^{\gamma} \tag{4.22}$$

Again in the case of the logarithmic utility function, the optimal consumption strategies are

$$c_i = \frac{W_0 \cdot \delta_i \cdot R_i}{\sum_{j=0}^{T} \delta_j \cdot s_j}, \quad i = 0, 1, ..., T \tag{4.23}$$

with the maximum value of the objective function

$$\max\left\{\sum_{i=0}^{T}\delta_i \cdot s_i \cdot \ln(c_i)\right\} = \sum_{i=0}^{T}\delta_i \cdot s_i \cdot \ln\left(\frac{W_0 \cdot \delta_i \cdot R_i}{\sum_{j=0}^{T}\delta_j \cdot s_j}\right) \qquad (4.24)$$

The formulas above enable analytical expression of annuity equivalent wealth AEW_0 due to access to perfect annuity markets PA in comparison with saving activities restricted to PB markets only (in Section 3 we were able to construct AEW_0 only graphically; see *Figure 3*). Obviously, it is sufficient to balance the maximum values of objective functions (4.14) for PB markets and (4.21) for PA markets using annuity equivalent wealth AEW_0 instead of initial capital W_0 for the case of PB markets:

$$\frac{AEW_0^{1-\gamma}}{1-\gamma}\cdot \Phi^{PB} = \frac{W_0^{1-\gamma}}{1-\gamma}\cdot \Phi^{PA} \qquad (4.25)$$

Hence one easily obtains the analytical formula for AEW_0 in the form

$$AEW_0 = W_0\left(\frac{\Phi^{PA}}{\Phi^{PB}}\right)^{1/(1-\gamma)} \qquad (4.26)$$

where Φ^{PB} is according to (4.15) and Φ^{PA} is according to (4.22).

4.3 Optimal Consumption in the Case of Classical Annuity Market CA

The consumptions $c_0, c_1, \ldots, c_T$ in the case of a classical annuity market with constant annuity payments $c_0 = c_1 = \ldots = c_T = c$ must fulfill

$$c = \frac{W_0}{\sum_{i=0}^{T} s_i R_i^{-1}} \qquad (4.27)$$

It gives the maximum value of the objective function

$$\sum_{i=0}^{T}\delta_i \cdot s_i \cdot \frac{c^{1-\gamma}}{1-\gamma} = \frac{W_0^{1-\gamma}}{1-\gamma}\cdot \Phi^{CA} \qquad (4.28)$$

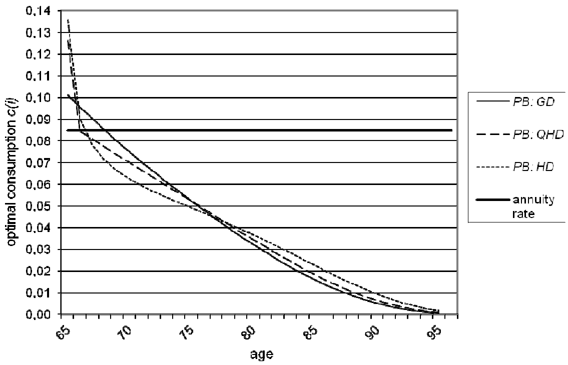
where

$$\Phi^{CA} = \frac{\sum_{i=0}^{T}\delta_i \cdot s_i}{\left(\sum_{i=0}^{T} s_i \cdot R_i^{-1}\right)^{1-\gamma}} \qquad (4.29)$$

In particular for the logarithmic utility function this maximum value is

$$\sum_{i=0}^{T}\delta_i \cdot s_i \cdot \ln(c) = \sum_{i=0}^{T}\delta_i \cdot s_i \cdot \ln\left(\frac{W_0}{\sum_{j=0}^{T} s_j \cdot R_j^{-1}}\right) \qquad (4.30)$$

Figure 5 Optimal Consumption Strategies {C_i} per Unit of Initial Capital W_0 in the Case of Perfect Bond Markets PB for the Objective Functions of the Types GD, QHD and HD for Males with Moderate Risk Aversion
(γ = 1; GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1; Life Tables for Males in the Czech Republic in 2010; pension age of 65; fixed annual interest rate of 2.5%, i.e. R = 1.025)



The analytical expression of AEW_0 due to access to classical annuity markets CA in comparison with saving activities restricted to PB markets is similar to that in (4.26)

$$AEW_0 = W_0 \left(\frac{\Phi^{CA}}{\Phi^{PB}} \right)^{1/(1-\gamma)} \tag{4.31}$$

where Φ^{PB} is according to (4.15) and Φ^{CA} is according to (4.29).

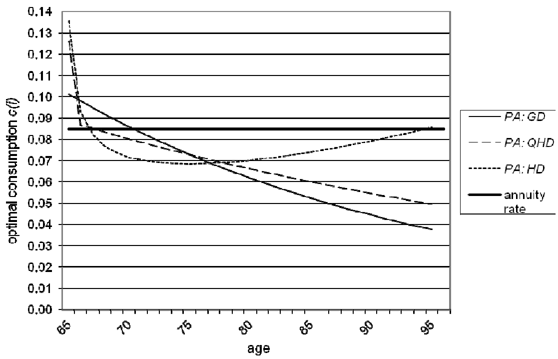
5. Numerical Results

In this section numerical values according to the analytical formulas derived in Section 4 are calculated for various (fixed) interest rates r and various levels of risk aversion γ.

Figure 5 plots the optimal consumption strategies $\{c_i\}$ per unit of initial capital W_0 in the case of perfect bond markets PB (see formula (4.13)) and *Figure 6* in the case of perfect annuity markets PA (see formula (4.20)). In both cases we have used the objective functions of the types GD, QHD and HD (see the discounting sequences in (4.5) with recommended coefficients) and the logarithmic utility function with moderate risk aversion ($RRA(c) = \gamma = 1$, see (3.2) and (3.3)); both figures concern male mortality in the Czech Republic in 2010 and suppose the pension age of 65 and fix the annual interest rate r to the upper limit of the so-called technical interest rate for commercial life insurance in the Czech Republic in 2013 (i.e. 2.5% or equivalently $R = 1.025$).

If the individual saves only without annuity instruments (see *Figure 5*) then initial consumption is relatively high but it decreases gradually (at a similar rate for all three discounting variants (4.5)) since at more advanced ages the individual has no further capital sources and must reduce consumption accordingly. This can be

Figure 6 Optimal Consumption Strategies {C_i} per Unit of Initial Capital W_0 in the Case of Perfect Annuity Markets PA for the Objective Functions of the Types GD, QHD and HD for Males with Moderate Risk Aversion
(γ = 1; GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1; Life Tables for Males in the Czech Republic in 2010; pension age of 65; fixed annual interest rate of 2.5%, i.e. R = 1.025)



compared with consumption in the case of a classical life annuity with a constant annuity rate of 0.084 9 = 8.49% of W_0 (see *Figure 5*).

If the individual has access to perfect annuity markets (see *Figure 6*), then the consumptions differ significantly according to the type of discounting. If using GD and QHD, then the individual is prone to consuming earlier since the interest rate r does not balance the individual's degree of impatience δ (e.g. $\delta = 0.957 < < 1/(1+r) = 1/1.025 = 0.976$). Conversely, if using HD, then the initial decrease of consumption can even turn into an increase (e.g. due to increasing expenses caused by deteriorating health).

Table 1 presents the values of annuity equivalent wealth AEW_0 (as a percentage of initial capital W_0) originating due to access to perfect annuity markets PA in comparison with saving activities in bond markets PB only (see formula (4.26)). Similarly *Table 2* presents the values of AEW_0 originating due to access to classical annuity markets CA again versus bond markets PB (see formula (4.31)). The objective functions of the types GD, QHD and HD and the utility function CRRA with different relative levels of risk aversion $RRA(c) = \gamma$ (see (3.3)) are used. The calculations are performed separately for males and females in the Czech Republic in 2010 and suppose the pension age of 65 and fix the annual interest rate r to 2.5% again.

For example, the value 120.6% from the upper left corner of *Table 1* (constructed for r = 2.5%) means that access to perfect annuity markets brings to a 65-year-old male with moderate risk aversion ($\gamma = 0.5$) capital improvement of 20.6%. This improvement due to annuitization is even more significant if the level of risk aversion is higher (even by 80% for $\gamma = 8$). On the other side, the type of discounting is nearly irrelevant.

The conclusions are similar in *Table 2,* which compares saving activities with access to classical annuity markets CA. Surprisingly the values in this table are only insignificantly lower than the values in *Table 1*. For example, the value 116.0% from the upper left corner of *Table 2* (constructed for $r = 2.5\%$) means that access to

Table 1 Annuity Equivalent Wealth AEW_0 (as a Percentage of Initial Capital W_0) Originating Due to Access to Perfect Annuity Markets PA in Comparison with Saving Activities in Bond Markets PB for the Objective Functions of the Types GD, QHD and HD for Males and Females with Different Risk Aversion $RRA = \gamma$
(GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1;
Life Tables for Males and Females in the Czech Republic in 2010;
pension of age 65; fixed annual interest rate of 2.5%, i.e. R = 1.025)

AEW (%) for PA versus PB: males						
Model	*Relative risk aversion* (*RRA* = *γ*)					
	0.5	*1.0*	*1.5*	*2.0*	*4.0*	*8.0*
GD	120.6	133.2	141.6	148.0	163.8	178.1
QHD	122.0	135.8	144.4	150.7	166.0	179.6
HD	125.1	140.9	149.7	156.0	170.4	182.7
AEW (%) for PA versus PB: females						
Model	*Relative risk aversion* (*RRA* = *γ*)					
	0.5	*1.0*	*1.5*	*2.0*	*4.0*	*8.0*
GD	113.5	122.5	128.6	133.2	144.7	155.3
QHD	115.2	124.9	131.0	135.5	146.6	156.5
HD	119.7	130.2	136.2	140.4	150.4	159.1

Table 2 Annuity Equivalent Wealth AEW_0 (as a Percentage of Initial Capital W_0) Originating Due to Access to Classical Annuity Markets CA in Comparison with Saving Activities in Bond Markets PB for the Objective Functions of the Types GD, QHD and HD for Males and Females with Different Risk Aversion $RRA = \gamma$
(GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1;
Life Tables for Males and Females in the Czech Republic in 2010;
pension age of 65; fixed annual interest rate of 2.5%, i.e. R = 1.025)

AEW (%) for CA versus PB: males						
Model	*Relative risk aversion* (*RRA* = *γ*)					
	0.5	*1.0*	*1.5*	*2.0*	*4.0*	*8.0*
GD	116.0	130.4	139.7	146.5	162.9	177.6
QHD	117.0	133.2	142.6	149.3	165.3	179.2
HD	119.0	137.8	147.6	154.4	169.6	182.3
AEW (%) for CA versus PB: females						
Model	*Relative risk aversion* (*RRA* = *γ*)					
	0.5	*1.0*	*1.5*	*2.0*	*4.0*	*8.0*
GD	108.4	119.5	126.5	131.6	143.8	154.8
QHD	110.5	122.5	129.4	134.3	145.9	156.2
HD	114.5	127.7	134.5	139.1	149.7	158.8

classical annuity markets brings to a 65-year-old male with moderate risk aversion ($\gamma = 0.5$) capital improvement of 16.0 % (which is comparable to the 20.6% improvement from *Table 1*). The reason is logical: the transfer from classical to perfect

Table 3 Expected Percentage of Initial Capital W_0 which Remains Unused by a Male Who Has No Access to Annuity Markets and Must Choose the Optimal Consumption Strategy on Perfect Bond Markets PB
(GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1;
Life Tables for Males in the Czech Republic in 2010; pension age of 65)

Expected percentage of W_0 (in %) for bond markets PB: males							
Model	*r*	*Relative risk aversion* (*RRA* = γ):					
		0.5	*1.0*	*1.5*	*2.0*	*4.0*	*8.0*
GD	0 %	11.8	20.6	26.7	31.2	41.5	49.5
	1 %	12.3	20.6	26.2	30.2	39.3	46.5
	2 %	12.9	20.6	25.6	29.2	37.3	43.6
	3 %	13.5	20.6	25.1	28.3	35.3	40.8
	4 %	14.2	20.6	24.6	27.4	33.5	38.2
QHD	0 %	11.8	21.5	27.8	32.3	42.3	50.1
	1 %	12.2	21.5	27.2	31.2	40.1	47.0
	2 %	12.9	21.5	26.6	30.2	38.0	44.1
	3 %	13.6	21.5	26.1	29.2	36.0	41.3
	4 %	14.3	21.5	25.5	28.2	34.1	38.6
HD	0 %	11.3	22.9	29.7	34.2	43.9	51.1
	1 %	12.1	22.9	29.0	33.1	41.6	48.0
	2 %	12.9	22.9	28.3	31.9	39.4	45.0
	3 %	13.7	22.9	27.7	30.8	37.3	42.1
	4 %	14.6	22.9	27.1	29.8	35.3	39.4

annuity markets has no impact on consumption restrictions; at most it only enables a change from a constant to variable consumption profile (actually, *Figure 6* indicates that the optimal consumption strategy in PA markets is not far from the constant consumption profile, so the benefits of pension flexibility are not very pronounced).

Finally, *Tables 3* and *4* strive to quantitatively demonstrate that it is really inconvenient when, for example, males or females aged 65 have no access to annuity markets and therefore they must rely upon their savings in old age. Strictly speaking, these tables offer (separately for males and females) the following values (for various interest rates and various levels of risk aversion)

$$100 \cdot \left(1 - \sum_{i=0}^{T} s_i \cdot R_i^{-1} \cdot c_i\right) = 100 \cdot \left(1 - \sum_{i=0}^{T} \frac{s_i \cdot c_i}{\prod_{j=1}^{i} \left(1 + r_j\right)^j}\right) (\%) \qquad (5.1)$$

The expression (5.1) presents the expected percentage of initial capital W_0, which remains unused by an individual who has no access to annuity markets and therefore looks for the optimal consumption strategy $\{c_i\}$ on perfect bond markets PB according to (4.13) using utility discounting from (4.5). One can see that these values can achieve even 50% of the initial capital (under strong risk aversion). Again, this

Table 4 Expected Percentage of Initial Capital W_0 which Remains Unused by a Female Who Has No Access to Annuity Markets and Must Choose the Optimal Consumption Strategy on Perfect Bond Markets PB
(GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1;
Life Tables for Females in the Czech Republic in 2010; pension age of 65)

Expected percentage of W_0 (in %) for bond markets PB: females							
Model	r	*Relative risk aversion* (*RRA* = γ):					
		0.5	*1.0*	*1.5*	*2.0*	*4.0*	*8.0*
GD	0 %	8.0	15.1	20.3	24.3	33.8	41.6
	1 %	8.5	15.1	19.8	23.3	31.6	38.5
	2 %	9.0	15.1	19.3	22.3	29.6	35.5
	3 %	9.5	15.1	18.8	21.4	27.6	32.7
	4 %	10.1	15.1	18.3	20.6	25.8	30.1
QHD	0 %	8.3	16.2	21.5	25.5	34.7	42.2
	1 %	8.9	16.2	20.9	24.4	32.5	39.0
	2 %	9.5	16.2	20.4	20.4	30.4	36.0
	3 %	10.2	16.2	19.8	22.4	28.4	33.2
	4 %	10.9	16.2	19.3	21.5	26.5	30.5
HD	0 %	8.9	18.2	23.8	27.7	36.5	43.3
	1 %	9.7	18.2	23.1	26.5	34.1	40.1
	2 %	10.5	18.2	22.5	25.4	31.9	37.0
	3 %	11.4	18.2	21.9	24.4	29.8	34.1
	4 %	12.3	18.2	21.3	23.3	27.9	31.4

confirms the fact that life annuities are effective instruments to ensure proper consumption in old age.

Example: In order to have a numerical idea, *Table 5* compares the optimal consumption strategies (using the logarithmic utility function) per initial capital of CZK 1,000,000 in the case of perfect bond markets PB and perfect annuity markets PA with a monthly life annuity for males using the pension age of 65. Even though initial consumption is relatively high without annuity instruments (one can perceive it as an initial lump sum consumed immediately), it decreases rapidly at more advanced ages. One can see the significant difference from the classical life annuity calculated applying the technical interest rate of 2.5%, which is common in commercial life insurance (Czech Republic, 2012-2014).

6. Conclusion: Annuity Puzzle

The conclusions resulting from the preceding analysis are straightforward: annuitization is a powerful instrument or one's economic assurance in old age, and this paper delivers various quantitative arguments for such a conclusion. On the other hand, although maximum annuitization should be the best of all scenarios, the common practice of preferring immediate consumption is quite different. In order to make this paper more practical, we will attempt in the conclusion to list some arguments addressing this strange annuity puzzle.

Table 5 Optimal Consumption Strategies per Initial Capital of CZK 1,000, 000 in the Case of Perfect Bond Markets PB and Perfect Annuity Markets PA Compared with the Monthly Life Annuity for Males with Moderate Risk Aversion

(γ = 1; GD: δ = 0.944; QHD: β = 0.7 and δ = 0.957; HD: η = 4 and ξ = 1;
Life Tables for Males in the Czech Republic in 2010;
pension age of 65; fixed annual interest rate of 2.5%, i.e. R = 1.025)

Age	**Monthly consumption (CZK)**						**Monthly life annuity (CZK)**
	PB			*PA*			
	GD	*QHD*	*HD*	*GD*	*QHD*	*HD*	
65	8442	10513	11294	8442	10513	11294	7075
66	7974	7047	7557	8169	7219	7742	7075
67	7521	6738	6519	7904	7081	6851	7075
68	7085	6435	5934	7648	6946	6405	7075
69	6658	6130	5524	7400	6813	6140	7075
70	6248	5831	5208	7160	6683	5969	7075
71	5851	5536	4946	6928	6556	5857	7075
72	5465	5242	4716	6704	6431	5785	7075
73	5089	4949	4504	6487	6308	5741	7075
74	4722	4655	4303	6277	6188	5719	7075
75	4359	4356	4101	6073	6070	5713	7075
76	4006	4058	3900	5876	5954	5722	7075
77	3662	3761	3697	5686	5840	5741	7075
78	3328	3466	3491	5502	5729	5770	7075
79	3005	3172	3278	5324	5620	5808	7075
80	2693	2882	3060	5151	5513	5853	7075
81	2393	2596	2835	4984	5407	5905	7075
82	2105	2315	2603	4823	5304	5963	7075
83	1832	2043	2366	4666	5203	6026	7075
84	1575	1780	2126	4515	5104	6095	7075
85	1335	1530	1885	4369	5006	6169	7075
86	1114	1294	1646	4227	4911	6247	7075
87	913	1076	1413	4090	4817	6330	7075
88	734	876	1190	3958	4725	6418	7075
89	576	698	980	3830	4635	6509	7075
90	441	542	787	3706	4547	6605	7075
91	328	408	614	3586	4460	6705	7075
92	236	298	464	3469	4375	6808	7075
93	164	210	338	3357	4292	6916	7075
94	109	141	236	3248	4210	7027	7075
95	69	91	157	3143	4129	7143	7075
96	42	55	99	3041	4051	7262	7075
97	24	32	59	2943	3973	7385	7075
98	12	17	33	2847	3898	7512	7075
99	6	8	17	2755	3823	7643	7075
100	3	4	8	2666	3750	7778	7075

Moreover, the pension puzzle is only one of the broader problems of economic reality concerning the pension topic which seem to be irrational at first sight: (*i*) for example, it is a known fact that moneyed seniors are not willing to plunder their savings. Such a weaker decumulation of savings at advanced ages can be partially caused by undeveloped annuity markets, as ordinary seniors are not usually capable of managing their assets to ensure a proper living; on the other hand, it is hard to explain the growing savings of seniors reported in developed countries (see, for example, Poterba, 1994). (*ii*) It seems also that the decumulation of savings (including consumption in old age) occurs mostly in jumps according to immediate needs, which contradicts the continuous flows following as optimal flows from utility models supposing access to annuity markets (see Section 4). Let's now provide some arguments that can help to explain the annuity puzzle at least in part. Some additional arguments, such as possible future inflation or political instability, are not mentioned here because they are well known (e.g. the former can be encountered by using the indexed bonds in the investment portfolios of pension funds).

6.1 Impact of Social Benefits

If individuals can rely on the existence of social benefits (e.g. the flat pension in the UK) then they must decide how much they will provide from their savings to raise these benefits. On the other hand, the existence of such savings is closely related to the volume of income during one's working life (see, for example, Pensions Commission, 2004).

6.2 Possible Investments in Risky Assets and Pension Deferral

The investment models discussed in this text do not involve the possibility to invest in assets that are riskier than bonds (let's summarize such assets under the denotation "stocks"). In this context, some authors (see, for example, Milevsky, 1998) suggest various pension strategies where initial capital W_0 is in the first instance invested in stocks and annuitization is postponed to an optimal age several years after the pension age. Since such an optimal age used to be much higher than the legislated pension age (in particular for females), it delivers another argument as to why life annuities are not so common in practice.

Let's demonstrate the previous consideration by means of a simple example. Let the average annual yield of bonds be 2% and of stocks 9%, with the risk margin for stocks being 4%. This means that a rational individual invests the corresponding savings in life annuities only if the annual yield from such an investment is at least 5%. Denoting the yearly survival probability in the given age as p, the investment in annuities will be obviously advantageous only if it holds that

$$\frac{1+r}{p} = \frac{1,02}{p} > 1,05 \tag{6.1}$$

where (in addition to the bond yield of $r = 2\%$) the mortality drag due to inequality $p < 1$ is also effective. Although the mortality drag is bundled with the risk of death, this risk is usually much lower than the risk of stocks. According to the Life Tables for males and females in the Czech Republic in 2010, inequality (6.1) occurs from age 68 for males and from age 75 for females. According to the pension strategy described above, one should defer annuitization until these ages, meanwhile investing in riskier assets (in practice this means that seniors should retain their money longer in productive pension funds).

6.3 Impact of Consumption Distribution during the Decumulation Period

In Section 4 we derived optimal consumption flows during the decumulation period of a pension. These flows are relatively smooth but can be different in practice:

(1) Seniors need irregularly higher amounts for investments related to their age (e.g. installation of a lift) or related to their state of health (e.g. hip replacement not covered by health insurance). Such financial flows are certainly not smooth and due to limited access to credits for seniors they can be very demanding with respect to the liquidity of seniors' assets.

(2) Consumption preferences change with age. At advanced ages, composition of these preferences shifts from culture, dining and travelling to care centers, health attendants and similar expenses.

(3) Moreover, the irregular character of seniors' spending is multiplied by the randomness and uncertainty of the corresponding changes: it is certain that the majority of seniors will have health problems but their timing and relevance is unknown in advance.

(4) There are suitable commercial insurance products which are suggested to cover the typical social and health problems of seniors. For example, long-term care (LTC) products (popular in Germany, Switzerland and UK) cover the disability to perform activities of daily living (ADL). However, such commercial products are usually very expensive and are accessible only for rich families.

6.4 Impact of Anti-selection

The impact of anti-selection concerning life annuities can be very strong. Such anti-selection follows from the simple fact that the purchase of a life annuity is reasonable only if the individual's life expectancy is long. This can have interesting consequences:

(1) If government pensions are guaranteed, then an individual with subjectively short life expectancy will not choose any annuitization of his or her savings due to the conviction that the institutional cover of pension is sufficient.

(2) In the case of moneyed seniors, it is difficult to distinguish whether their pension behavior is influenced by their wealth or whether they take advantage of anti-selection.

6.5 Impact of Marketing

If pension products are offered by trained sales representatives, the presentation of the products can play an important role, including unexpected psychological effects, e.g. different feelings of satisfaction expressed by clients with defined benefit (DB) and defined contribution (DC) pension plans (see, for example, Drinkwater and Sondergeld, 2004).

REFERENCES

Angeletos G-M et al. (2001): The Hyperbolic Consumption Model: Calibration, Simulation, and Empirical Evaluation. *Journal of Economic Perspectives*, 13:47–68.

Arrow KJ (1965): Aspects of a Theory of Risk Bearing. *Yrjo Jahnsson Lectures*, Helsinki.

Backus D, Routledge B, Zin S (2004): Exotic Preferences. *NBER Working Paper*, no. 10597.

Booth P et al. (1999): *Modern Actuarial Theory and Practice*. Chapman and Hall/CRC, London.

Cannon E, Tonks I (2008): *Annuity Market*. Oxford University Press, Oxford.

Cipra T (2010): *Financial and Insurance Formulas*. Physica Verlag/Springer, Heidelberg, Dordrecht, London, New York.

Cipra T (2012): *Pension: Quantitative Approach*. Ekopress, Prague (in Czech).

Davidoff T, Brown JR, Diamond PA (2005): Annuities and Economic Welfare. *American Economic Review*, 95:1573–90.

Drinkwater M, Sondergeld ET (2004): Perceptions of Mortality Risk: Implications for Annuities. In: Mitchell OP, Utkus SP (eds.): *Pension Design and Structure: New Lessons from Behavioral Finance*. Oxford University Press, Oxford.

Epstein LG, Zin SE (1989): Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica*, 57:937–69.

HMSO, Pensions Commission (2004): *Pensions: Challenges and Choices: The First Report of the Pensions Commission*. HMSO, London.

Koller M (2000): *Stochastische Modelle in der Lebensversicherung*. Springer, Berlin.

Lee EM (1986): *An Introduction to Pension Schemes*. The Institute of Actuaries and the Faculty of the Actuaries, London.

McGill DM (1975): *Fundamentals of Private Pensions*. Irwin, Homewood, Illinois.

Milevsky MA (1998): Optimal Asset Allocation towards the End of the Life Cycle: To Annuitize or not to Annuitize? The *Journal of Risk and Insurance,* 65:401–26.

Milevsky MA (2006): *The Calculus of Retirement Income.* Cambridge University Press, Cambridge.

Mitchell OS et al. (1999): New Evidence on the Money's Worth of Individual Annuities. *American Economic Review*, 89:1299–1318.

Poterba JM (1994): *International Comparisons of Household Saving*. University of Chicago Press, Chicago.

Pratt JW (1964): Risk Aversion in the Small and in the Large. *Econometrica*, 32:122–36.

Rubinstein A (2003): Economic and Psychology? The Case of Hyperbolic Discounting. *International Economic Review*, 44:1207–16.

Sheshinski E (2008): *The Economic Theory of Annuities.* Princeton University Press, Princeton.

Winklevoss HE (1977): *Pension Mathematics with Numerical Illustrations.* Irwin, Homewood, Illinois.

Yaari M (1965): Uncertain Lifetime, Life Assurance, and the Theory of the Consumer. *Review of Economic Studies*, 32:137–50.