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Market Application of the Fuzzy-Stochastic Approach in the Heston Option Pricing Model*

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Abstract
The present study analyzes the extra insights that option pricing models may achieve when uncertainty about parameters is modeled through fuzzy numbers. Specifically, we consider the Heston stochastic volatility model, which assumes that stock price changes and their instantaneous variance evolve as a bivariate, possibly correlated, diffusive process. The original Heston model provides a quasi-closed formula for the pricing of several derivative products such as European options. By applying the fuzzy extension principle, we generalize the model to the case of fuzzy parameters; given their shape we are able to derive the membership of the fuzzy price of a European option. Finally, to understand the extent to which our approach might be useful in practice, we give a numerical illustration of our procedure on the S&P 500 and VIX indexes. As a by-product of our research, a simple estimation method is introduced to obtain (crisp) parameters in the Heston model under the risk-neutral measure and applied in the sequel of the paper to obtain alternative shapes for the fuzzy parameters of the model.

1. Introduction

In recent research several studies have been developed in order to handle, in a proper way, the intrinsic uncertainty ever present in financial and economic models. Many authors, such as Hryniewicz (2010) and Zadeh (2008), have argued that the mathematical theory of fuzzy numbers is the correct description of vagueness and imprecision; in Zadeh (2008) it is also underlined that fuzziness exists in many fields, especially in human sciences such as economics, and the application of fuzzy mathematics can provide rigorous results.

The origin of the mathematical theory of fuzzy numbers is essentially due to Zadeh (Zadeh, 1965), but many results in this field were achieved by Dubois and Prade (1980, 1993).

Most of the financial models in derivative pricing theory describe market uncertainty through the stochastic evolution of the price of the underlying assets, where some constant parameters are assigned. We are confident that extra value may be added to these stochastic models by jointly considering the uncertainty of the evolution and the vagueness of the parameters involved, which may be modeled through fuzzy numbers. By assuming that parameters are fuzzy, it is possible to reflect in the shape of their membership function both objective features and personal beliefs about the behavior of the parameters themselves.

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We follow the stream of research introduced by Zmeškal, who first introduced the fuzzy-stochastic approach and so-called hybrid models (see, for instance, Zmeškal, 2001, 2010). The first approach to the financial applications of such models is given in Guerra, Sorini, and Stefanini (2011), where the advantages of the fuzzy-stochastic approach are investigated in the Black and Scholes environment. In the quoted paper, the key parameters of the model are the volatility, the risk-free interest rate, and the price of the underlying asset. It is observed by the authors that the fuzzy price for the call option seems not to be heavily influenced by the shape of the fuzzy volatility para-meter.

Market practitioners, however, claim that volatility behavior is crucial in the market; a central issue is the unsatisfactory hypothesis of constant volatility through time and its inconsistency with *stylized facts* observed in financial data. In order to generalize the Black and Scholes constant volatility assumption, a vast literature is devoted to volatility modeling and volatility forecasting.

The motivation for this paper is based on our conviction that, given the relevance of the volatility parameter/variable in the market, the possible use of fuzzy theory in volatility modeling deserves deeper investigation.

Volatility models usually assume that volatility is itself a stochastic process; among others, the Heston model (Heston, 1993) is greatly appreciated due to the availability of a closed formula for the price of European options and other derivatives.

In this paper we generalize the Heston setting by assuming that the parameters are modeled as fuzzy numbers and we show how their vagueness affects the final option price, which is also obtained as a fuzzy number. The analysis is performed for several choices of the shapes of the membership functions of the parameters. In Figà-Talamanca and Guerra (2009) and Figà-Talamanca, Guerra, Sorini, and Stefanini (2010), we present preliminary studies on the same argument.

Stochastic volatility modeling in a fuzzy scenario has been previously addressed in the literature. For instance, in Thavaneswaran, Thiagarajah, and Appadoo (2007), the idea of fuzzy parameters is introduced in a discrete stochastic setting and fuzzy numbers are used to incorporate volatility variability. In Hung (2009) and Thavaneswaran, Appadoo, and Paseka (2009), Generalized Autoregression Conditional Heteroskedasticity (GARCH) discrete models are analyzed in a fuzzy context. In the former contribution the author modifies the threshold values for a positive/ /negative information distinction with a fuzzy rule and many empirical investigations are reported in order to validate the method. In the latter one, the authors study the centered moments and kurtosis for a class of FCA (Fuzzy Coefficient Autoregressive) and FCV (Fuzzy Coefficient Volatility) models.

Finally, in Swishchuk, Ware, and Li (2008), the authors also investigate the Heston model and obtain a fuzzy option price in the same context as ours; however, they assume that the instantaneous (local) volatility is itself a fuzzy number and derive its membership by transforming the probability distribution of the instantaneous variance process to its possibility distribution through the method described in Dubois, Prade, and Sandri (1993). The non-linear fuzzy PDE is then used to price European options.

On the contrary, we retain the original dynamics specification for the instantaneous variance and assign a membership function to the model parameters;

the fuzzy price for the option is then obtained by applying the fuzzy extension principle.

Note that fuzzy option prices may prove useful in real markets with frictions; in fact, the range of the support (or a different *α-cut*) of the fuzzy option price may be considered as a measure of the bid-ask spread.

Another contribution of this work is the idea of applying a rolling estimation method for crisp model parameters in order to design fuzzy parameter shapes which are consistent with empirical market observations. We point out that estimation of the Heston model is still a subject of ongoing research and has stimulated a debate on whether to use stock or derivative market data. So, as a by-product of our research, we also introduce a simple method to derive the Heston model parameters, based on both stock and derivatives information. More precisely, the suggested procedure is based on joint observations of the stock index and of its volatility index (the value of which depends on the price of traded options on the index itself). In our numerical illustration the S&P 500 stock index and the VIX volatility index are considered. While we apply the rolling approach to the Heston model, and considering our estimation method, it is worth noting that the same idea may be applied to any model as well as for any estimation method.

The paper is divided into five sections. After the introduction, in Section 2 we give a brief description of the Heston model and we introduce the basic elements of the theory of fuzzy numbers. The third section is devoted to the introduction of our simple estimation procedure for Heston (crisp) parameters and to describe how it is used for the empirical construction of several possible supports and shapes for the fuzzy parameters. In Section 4 a numerical experiment is outlined in order to analyze how fuzzy option prices are obtained in our setting. A final section, devoted to concluding remarks, underlines the relevance of the results obtained and traces some paths for possible future research.

2.1 The Heston Model

The Heston model (Heston, 1993) is a benchmark among stochastic volatility models due to the availability of a closed formula for the price of several derivative securities. In particular, the price at time t of a European call option with maturity T and strike price X is given by the following formula:

$$C_t(T,X) = S_t\Psi_1 - Xe^{-r(T-t)}\Psi_2 \tag{1}$$

where r is the market risk-free rate, assumed to remain constant until maturity.

The quantities Ψ_1 and Ψ_2 represent the probability that the option will be exercised at maturity with respect to different probability measures and are obtained through inversion of a Fourier transform. More precisely, for $j = 1, 2$ we have

$$\Psi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\ln X} F_j\left(\ln S_t, v, T-t; u\right)}{iu}\right] du$$

with

$$F_j(x,v,T-t;u) = \exp\left\{A_j(T-t,u) + B_j(T-t)v + iux\right\}$$

Defining $\tau = T - t$, the exact expressions of the functions A_j and B_j are

$$A_j(\tau,u) = rui\tau + \frac{a}{c^2}\left\{\left(b_j - \rho cui + d_j\right)\tau - 2\ln\left[\frac{1-g_j e^{d_j\tau}}{1-g_j}\right]\right\}$$

$$B_j(\tau,u) = \frac{b_j - \rho cui + d_j}{c^2}\left[\frac{1-e^{d_j\tau}}{1-g_j e^{d_j\tau}}\right]$$

where

$$g_j = g_j(u) = \frac{b_j - \rho cui + d_j}{b_j - \rho cui - d_j}$$

$$d_j = d_j(u) = \sqrt{\left(\rho cui - b_j\right)^2 + c^2u^2}$$

and

$$b_1 = \kappa^* - \rho c$$
$$b_2 = \kappa^*$$
$$a = \kappa^*\theta^*$$

Hence, we have

$$C_t(T,X) = C_t\left(\tau, X, S_t, r, \rho, \kappa^*, \theta^*, c, v\right) \tag{2}$$

where κ^*, θ^*, c are the parameters appearing in the joint dynamics of the price S_t (at time *t*) of the underlying stock and its local variance V_t. Such dynamics are specified, under the so-called risk-neutral probability measure, by the following bidimensional stochastic equation:

$$d\log S_t = \sqrt{V_t}\,dB_t$$
$$dV_t = \kappa^*\left(\theta^* - V_t\right)dt + c\sqrt{V_t}\,dW_t$$

where *(B,W)* is a possibly correlated Brownian motion with $\langle dB_t, dW_t\rangle = \rho dt$ (see Heston, 1993, for further details). The instantaneous variance V_t is thus modeled by a mean-reverting process where parameter θ^* represents the long-run mean variance, κ^* is the speed of return to the long-run mean θ^*, and c is the so-called *volatility of volatility* parameter.

2.2 Fuzzy Numbers and the Extension Principle

In order to describe the methodology applied, we recall some preliminaries about fuzzy numbers. A fuzzy number $u = \langle a^-, a, a^+\rangle_{\mathrm{L,R}}$ is usually specified by its core $a \in \mathrm{R}$ and a membership function $\mu : \mathrm{R} \to [0,1]$, with support in $[a^-, a^+]$

defined as

$$\mu(x)=\begin{cases}\mathrm{L}(x) & \text{if} \quad a^{-}\le x\le a\\ \mathrm{R}(x) & \text{if} \quad a\le x\le a^{+} \ \text{ for } x\in \mathrm{R}\\ 0 & \text{otherwise}\end{cases}$$

where $\mathrm{L}(x)$ is an increasing function with $\mathrm{L}(a^{-})=0$, $\mathrm{L}(a)=1$ and $\mathrm{R}(x)$ is a decreasing function with $\mathrm{R}(a)=1$, $\mathrm{R}(a^{+})=0$. Functions L(.) and R(.) are the left and right shape functions of *u*, and they are assumed to be differentiable. For values of $\alpha\in[0,1]$, the *α-cuts* are defined to be the compact intervals $[u]_\alpha=\{x\,|\,\mu(x)\ge\alpha\}$, which are "nested" closed intervals.

For our purposes it is more convenient to specify fuzzy parameters with the Lower-Upper (LU) representation introduced in Stefanini, Sorini, and Guerra (2006); we briefly recall that an LU-fuzzy number *u* is determined by any pair $u=\left(u^{-},u^{+}\right)$ of functions $u^{-},u^{+}\ :\ \left[0,1\right]\to\mathrm{R}$, defining the end-points of the *α-cuts*, satisfying some conditions, and possessing non-empty and compact *α-cuts* of the form $[u]_\alpha=\left[u_\alpha^{-},u_\alpha^{+}\right]\subset\mathrm{R}$.

In the LU representation the support of *u* is the interval $\left[u_0^{-},u_0^{+}\right]$ the core is $\left[u_1^{-},u_1^{+}\right]$, and the lower and upper branches $u_{(.)}^{-}$ and $u_{(.)}^{+}$ are continuous invertible functions defining the membership function $\mu_u(.)$ as two continuous branches, the left being the increasing inverse of $u_{(.)}^{-}$ on $\left[u_0^{-},u_1^{-}\right]$ and the right the decreasing inverse of $u_{(.)}^{+}$ on $\left[u_1^{+},u_0^{+}\right]$.

The two monotonic branches u_α^{-} and u_α^{+} are parametrized over a decomposition of the interval $\left[0,1\right]$ into *N* sub-intervals $\left[\alpha_{i-1},\alpha_i\right]$ for *j* = 1,2,…,*N*. For each decomposition, 4(*N*+1) parameters are required:

$$u=\left(\alpha_i;u_i^{-},\delta u_i^{-},u_i^{+},\delta u_i^{+}\right)_{i=0,1,\ldots,N}$$

satisfying the following conditions:

$$u_0^{-}\le u_1^{-}\le\ldots\le u_N^{-}\le u_N^{+}\le u_{N-1}^{+}\le\ldots\le u_0^{+}\ \text{(data)}$$

$$\delta u_i^{-}\ge 0,\delta u_i^{+}\le 0\ \text{(slopes)}$$

The simplest representation is obtained on the trivial decomposition of the interval $[0,1]$, with *N* = 1 (without internal points) and $\alpha_0=0,\alpha_1=1$. In this simple case, *u* can be represented by a vector of eight components

$$u=(u_0^{-},\delta u_0^{-},u_0^{+},\delta u_0^{+};u_1^{-},\delta u_1^{-},u_1^{+},\delta u_1^{+})$$

where $u_0^{-},\delta u_0^{-},u_1^{-},\delta u_1^{-}$ are used for the lower branch u_α^{-}, and $u_0^{+},\delta u_0^{+},u_1^{+},\delta u_1^{+}$ for the upper branch u_α^{+}.

When managing functions of real variables, the fuzzy extension has to be the result of correct application of the extension principle. Given a function $y = f(x_1, x_2, ..., x_n)$ of *n* real variables $x_1, x_2, ..., x_n$, its fuzzy extension is obtained to evaluate the effect of uncertainty on the x_j modeled by the corresponding fuzzy number u_j, i.e., for each level α by the interval $\left[u_{j,\alpha}^-, u_{j,\alpha}^+\right]$ giving the possible values of x_j for that level. If $v = f(u_1, u_2, ..., u_n)$ denotes the fuzzy extension of a continuous function *f* in *n* variables, for each level α the resulting interval $\left[v_\alpha^-, v_\alpha^+\right]$ represents the propagation of uncertainty from the variables x_j to the variable *y*. In particular, if the uncertainty on the original variables is also modeled by linear fuzzy numbers, the obtained v is still a fuzzy number starting from a single value (at level $\alpha = 1$) to the most uncertain interval (at level $\alpha = 0$), but it loses the linearity property. It follows that parametric representation is also necessary when the input variables are triangular fuzzy numbers in order to apply the extension principle and to represent the non-linear output fuzzy numbers.

In practice, to obtain the fuzzy extension of *f* to normal upper semi-continuous fuzzy intervals, we have to compute the *α-cuts* $[v_\alpha^-, v_\alpha^+]$ of v, defined as the images of the *α-cuts* of $(u_1, u_2, ..., u_n)$ that are then obtained by solving the following box-constrained optimization problems for $\alpha \in [0,1]$:

$$(EP)_\alpha \quad : \quad \begin{cases} v_\alpha^- = \min\left\{ f(x_1, x_2, ..., x_n) \mid x_k \in \left[u_{k,\alpha}^-, u_{k,\alpha}^+\right],\ k = 1, ..., n \right\} \\ v_\alpha^+ = \max\left\{ f(x_1, x_2, ..., x_n) \mid x_k \in \left[u_{k,\alpha}^-, u_{k,\alpha}^+\right],\ k = 1, ..., n \right\} \end{cases}$$

Only in simple cases can the optimization problems above be solved analytically. In general, the solution is difficult and computationally expensive to find, as, for each $\alpha \in [0,1]$, the global solutions of two non-linear programming problems are required.

2.3 The Fuzzy Heston Pricing Formula

It is worth noting that the option value obtained through the above-mentioned pricing formula (1) can only be computed when we have complete information on the parameters. This is hardly the case when dealing with real data applications. Of course, the parameters are not known and should be estimated; furthermore, as already noted in the introduction, many procedures have been introduced in the statistical and financial literature for the estimation of Heston model parameters and it is a subject of debate whether the estimates should be obtained by using pure statistical methods based only on a time series of the stock price (considering the instantaneous variance V_t as a latent variable) or calibration methods based essentially on market option prices (some highlights can be found in Figà-Talamanca and Guerra, 2006). In addition, some authors have suggested using proxies for process V_t, such as the realized variance (see, for instance, Bollerslev and Zhou, 2002). It is hard to establish the best solution, and even though we were able to choose one of the methods, statistical estimation is imprecise by definition, not to mention that parameter estimates change over time. In Benhamou, Gobet, and Miri (2009), an effort is made to extend the Heston model to the case of time-dependent parameters, but this drastically reduces the analytical tractability of the model.

To cope with the imprecision of the parameters, which is the final aim of this analysis, retaining both the parsimony and analytical tractability characterizing the Heston model, we generalize the model by assuming the fundamental parameters to be fuzzy numbers.

More precisely, we set

$$\kappa^* = \left\langle \kappa^-, \kappa, \kappa \right\rangle_{\mathrm{L,R}}, \quad \theta^* = \left\langle \theta^-, \theta, \theta^+ \right\rangle_{\mathrm{L,R}}$$

$$c^* = \left\langle c^-, c, c^+ \right\rangle_{\mathrm{L,R}}, \quad \rho = \left\langle \rho^-, \rho^c, \rho^+ \right\rangle$$

and $\nu = \left\langle \nu^-, \nu^c, \nu^+ \right\rangle$ and we assign the corresponding membership functions $\mu(\kappa^*)$, $\mu(\theta^*)$, $\mu(c^*)$, $\mu(\rho)$, $\mu(\nu)$. As soon as information on the parameters is modeled by fuzzy numbers, the value of the call option in (2) also becomes fuzzy and may be represented by its membership as well as by its *α-cuts* $\left[C_\alpha^{*^-}, C_\alpha^{*^+} \right]$ for all degrees α of possibility. The maximum uncertainty corresponds to the support at $\alpha = 0$. This membership function and the corresponding *α-cuts* are obtained by applying the fuzzy extension of function $C_t(T, X)$ in (2) in a rigorous way. The shape of the membership obtained for the option price serves as a measure of imprecision propagation from the parameters to the call option value. It is worth noting that $C_t(T, X)$ is highly non-linear in the parameters, making the computation of the corresponding fuzzy-valued function a tricky step. To reduce the computational burden we implement the multiple population differential evolution (DE) algorithm, which is designed to compute the values and the slopes of the LU representation of the fuzzy extensions of (2).

3. Derivation of the Fuzzy Support for the Model Parameters

To obtain the fuzzy support of the model parameters and properly assign the memberships of the parameters we proceed in two steps. First, we introduce a simple estimation technique for the Heston model parameter, under risk-neutral dynamics; then, we build the fuzzy support by applying the estimation procedure on rolling (moving) windows of the data. Our procedure provides estimates for the risk-neutral parameters; it is based on observations for the stock index price and for the volatility index value. The underlying idea is to read the volatility index as a proxy for the integrated volatility under the risk-neutral probability. This latter approach is a statistical alternative to the calibration of parameters obtained through the minimization of the distance between theoretical and market option prices (see Cont and Kokholm, 2009, and Sepp, 2008). We detail the procedure by considering the S&P 500 stock market index and its volatility index, the VIX.

The VIX is a volatility index computed on the basis of prices for options on the S&P 500 index (SPX options). Denote

$$v_t^T = \frac{2}{T-t}\sum_i \frac{\Delta K_i}{K_i^2} e^{r_t^T(T-t)} Q(K_i,T;t) - \frac{1}{T-t}\left(\frac{F_t}{K_0}-1\right)^2 \tag{3}$$

where T is the time to maturity of all considered options, F_t is the forward index level at time t (derived from index option prices), K_0 is the first strike below F_t, K_i is the strike price of the i-th out-of-the money option and $\Delta K_i = (K_{i+1} - K_{i-1})/2$, r_t^T is the risk free rate at time t for a bond with maturity T, and $Q(K_i,T;t)$ is the midpoint of the bid-ask values for the option with strike K_i. Then, the value of the VIX at time t is given by the following formula:

$$VIX_t = 100 * \sqrt{\frac{365}{30}\left((T_1-t)v_t^{T_1}\frac{N_{T_2}-30}{N_{T_2}-N_{T_1}} + (T_2-t)v_t^{T_2}\frac{30-N_{T_1}}{N_{T_2}-N_{T_1}}\right)}$$

i.e., the VIX value is finally obtained as a weighted mean of the output of formula (3) for values T_1 and T_2 corresponding to near and next term maturity.

The options to be considered in the calculation of the VIX are out-of-the-money calls and puts with non-zero bid prices, centered around the at-the-money strike price K_0. Full details on the VIX computation are available in the VIX white paper The *CBOE Volatility Index, 2009*.

As mentioned above, we use the VIX volatility index as a proxy for the square root of the expectation (under the risk-neutral measure) of the annualized integrated variance over a one-month horizon.

Specifically, if we define the integrated variance process $I(t,\tau)$ as

$$I(t,\tau) = \frac{1}{\tau}\int_t^{t+\tau} V_s ds$$

we assume that

$$(VIX/100)^2 = E^Q\left[I\left(t,\frac{30}{365}\right)\right]$$

Hence, the value of the square VIX/100 index at time t is assumed to be the sample observation, under the risk-neutral measure, of the process $I(t,\tau)$ with $\tau = \frac{30}{365}$.

The heuristic motivation for this choice is given in Sepp (2008). A detailed analysis of this assumption is beyond the scope of our work; however, the interested reader can find a theoretical justification in Cont and Kokholm (2009).

In addition, we take advantage of the fact that the moments of the integrated variance under the Heston dynamics specification can be derived as a function of the parameters. We know from the results in the literature (see Figà-Talamanca, 2009, for instance) that $I(t,\tau)$ is a stationary process for a fixed value of τ, for which

$$E\left[I(t,\tau)\right] = E\left[I(0,\tau)\right] = \theta^*$$

$$Cov\left[I(t,\tau), I(t+h,\tau)\right] = Var\left(V_0\right)\frac{\left(1-e^{-\kappa^*\tau}\right)^2}{\kappa^{*2}\tau^2}e^{-(h-1)\kappa^*\tau},\ h \geq 1$$

where expectations are computed with respect to the risk-neutral probability measure.

Hence

$$\frac{Cov\left[I(t,\tau), I(t+1,\tau)\right]}{Cov\left[I(t,\tau), I(t+h,\tau)\right]} = e^{(h-1)\kappa^*\tau}$$

Given m observations of the VIX index, namely, $VIX_1, VIX_2, ..., VIX_m$, and defining $\hat{I}_t = \left(\frac{VIX_t}{100}\right)^2$ we thus suggest the following estimates, respectively, for the long-run mean and mean reversion speed:

$$\hat{\theta}^* = \frac{1}{m}\sum_{t=1}^{m}\hat{I}_t$$

$$\hat{\kappa}^* = \frac{1}{h-1}\log\frac{\gamma\left(\hat{I}_t,1\right)}{\gamma\left(\hat{I}_t,h\right)}$$

where $\gamma\left(\hat{I}_t,h\right) = Cov\left[\hat{I}\left(t,\tau\right), \hat{I}\left(t+h,\tau\right)\right]$ is the empirical autocovariance function of $\hat{I}_t$ at lag h.

Once $\hat{\theta}^*$, $\hat{\kappa}^*$ are computed we can estimate $Var(V_0)$ by writing

$$\gamma\left(\hat{I}_t,1\right) = \widehat{Var}\left(V_0\right)\frac{\left(1-e^{-\hat{\kappa}^*\tau}\right)^2}{\hat{\kappa}^{*2}\tau^2}$$

from which

$$\widehat{Var}\left(V_0\right) = \gamma\left(\hat{I}_t,1\right)\frac{\hat{\kappa}^{*2}\tau^2}{\left(1-e^{-\hat{\kappa}^*\tau}\right)^2}$$

Since, under the Heston model assumptions, $Var\left(V_0\right) = \frac{\theta^* c^2}{2\kappa^*}$, we may compute parameter $\hat{c}$ as

$$\hat{c}^2 = \frac{2\hat{\kappa}^*}{\hat{\theta}^*}\widehat{Var}\left(V_0\right)$$

Parameter ρ is simply derived as the sample correlation between the S&P 500 index excess returns and the $\hat{I}_t$ time-series:

$$\hat{\rho} = \frac{\sum_{i,j=1}^{m} \left(SP_i - \overline{SP}\right)\left(\hat{I}_j - \bar{\hat{I}}\right)}{\sqrt{\sum_{i,j=1}^{m} \left(SP_i - \overline{SP}\right)^2 \left(\hat{I}_j - \bar{\hat{I}}\right)^2}}$$

where

$$\overline{SP} = \frac{1}{m}\sum_{i=1}^{m} SP_i$$

$$\bar{\hat{I}} = \frac{1}{m}\sum_{i=1}^{m} \hat{I}_i$$

are the sample means of the S&P 500 excess returns and of the proxy for the integrated variance, respectively.

3.1 Choice of the Fuzzy Support

The support of the fuzzy parameters may be constructed—when a dataset of n observations is available—by rolling the estimation procedure on moving windows of length m so that $k = n - m$ estimates are available for each parameter. In our numerical illustration we are given a dataset for the S&P 500 index and its volatility index (the VIX Index) from January 1990 to October 22, 2010 for a total number of $n = 5{,}244$ daily observations. The estimation is repeated $k = 3{,}500$ times for (overlapping) moving windows of length $m = 1{,}744$ of the VIX index; the estimated values are reported in *Figure 1* for each parameter of the model. Considering the time span, the estimates show good stability performance. Of course, the most volatile of the parameters is the correlation, which is the only estimate relying on both the S&P 500 return and the VIX series.

Given these outcomes we suggest several possible ways to obtain the fuzzy support for each parameter; in all cases the crisp value is given by the median estimated value. We consider linear fuzzy numbers with the support having the extremes obtained according to a percentage variation p of the crisp value. If we denote by $\bar{x}$ the median of a generic parameter, the support is the closed interval $[\bar{x}*(1-p), \bar{x}, \bar{x}*(1+p)]$ and is, by construction, a centered interval on $\bar{x}$.

In *Figure 2* the histogram is reported for each of the four parameters. It is clear from the figures that the estimated values are asymmetric with respect to their median value. In order to take into account this asymmetry we use the empirical distribution of the parameters for an alternative definition of the parameter support.

We therefore define the fuzzy support of parameter x as $[l(x), h(x)]$, where $l(x)$ is a low percentile ($l\%$) and $h(x)$ is a high percentile ($h\%$) of the estimated values for x. Typical examples would be $l = 10$ and $h = 90$ or $l = 25$ and $h = 75$. As a natural generalization we also consider trapezoidal supports of the type $\left[l(x), m_1(x), m_2(x), h(x)\right]$ considering percentiles around the median ($m_1 = 45$, $m_2 = 55$). Note that this approach to the construction of *market data consistent* fuzzy shapes may be used in different model settings as well as for different estimation procedures.

Figure 1 S&P500 Index and VIX volatility: Parameters Estimated Values from the Beginning of 2000 to the End 2005 (from top left, clockwise we have *κ∗, θ∗, c, ρ*)

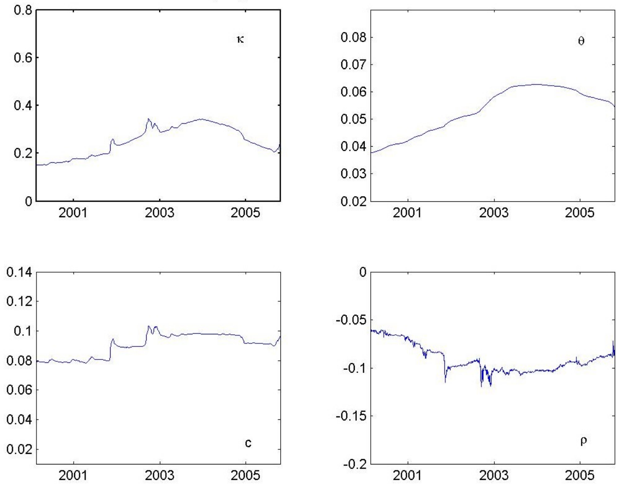
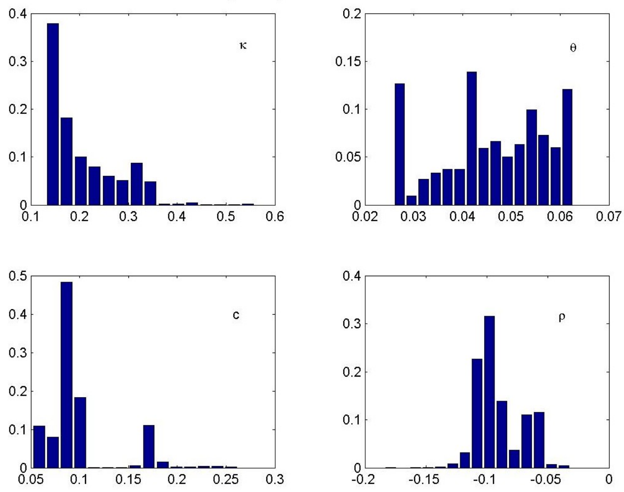


Figure 2 S&P500 Index and VIX volatility: Histograms of Parameters Estimated Values from the Beginning of 2000 to the End 2005 (from top left, clockwise we have *κ∗, θ∗, c, ρ*)



4. Numerical Illustration

We are given a dataset of the VIX index values from January 1990 to October 22, 2010 for a total number of $n = 5{,}244$ daily observations. In order to derive the support of the fuzzy parameters the estimation procedure described above is repeated $k = 3{,}500$ times for (overlapping) moving windows of length $m = 1{,}744$. From the 3,500 values obtained we derive the percentiles reported in *Table 1*.

Table 1

	$\hat{\theta}^*$	$\hat{\kappa}^*$	$\hat{c}$	$\hat{\rho}$
Min.	0.02589	0.1306	0.05147	-0.18419
10^{th}	0.02735	0.1384	0.06340	-0.10989
25^{th}	0.03986	0.1468	0.08043	-0.10319
45^{th}	0.04479	0.1662	0.08281	-0.09747
50^{th}	0.04671	0.1747	0.08443	-0.09489
55^{th}	0.04891	0.1838	0.08818	-0.09364
75^{th}	0.05549	0.2491	0.09761	-0.07522
90^{th}	0.06127	0.3209	0.16598	-0.06089
Max.	0.06273	0.5593	0.26232	-0.03160

In our experiment, denoting by $\{x_i\}_i$ the time series of estimated *d* values for the parameter *x* and by $q\left(\{x_i\}_i, n\right)$ its *n*-th percentile, the following specifications for the support of each fuzzy parameter *x* are developed:

– $x_1^- = x_1^+ = \overline{x}, x_0^- = x_1^- * 0.20, \quad x_0^+ = x_1^- * 1.80$, providing a symmetric triangular fuzzy number ($p = 0.8$);

– $x_1^- = x_1^+ = \overline{x}, \quad x_0^- = x_1^- * 0.60, \quad x_0^+ = x_1^- * 1.40$, providing a symmetric triangular fuzzy number ($p = 0.4$);

– $x_0^- = q\left(\{x_i\}_i, 10\right), \quad x_0^+ = q\left(\{x_i\}_i, 90\right), \quad x_1^- = x_1^+ = \overline{x},$ providing an asymmetric triangular fuzzy number;

– $x_0^- = q\left(\{x_i\}_i, 25\right), \quad x_0^+ = q\left(\{x_i\}_i, 75\right), \quad x_1^- = x_1^+ = \overline{x},$ providing an asymmetric triangular fuzzy number;

– $x_0^- = q\left(\{x_i\}_i, 10\right), \quad x_0^+ = q\left(\{x_i\}_i, 90\right), \quad x_1^- = q\left(\{x_i\}_i, 45\right), \quad x_1^+ = q\left(\{x_i\}_i, 55\right),$ providing a trapezoidal fuzzy number;

– $x_0^- = q\left(\{x_i\}_i, 25\right), \quad x_0^+ = q\left(\{x_i\}_i, 75\right), \quad x_1^- = q\left(\{x_i\}_i, 45\right), \quad x_1^+ = q\left(\{x_i\}_i, 55\right),$ providing a trapezoidal fuzzy number.

In *Figure 3* we plot, as an example, the six different shapes for parameter κ^* according to the definitions given for the fuzzy support in the above list. It is worth noting that the empirical distribution of κ^* is very skewed; while the higher extremes of the support are similar for cases 1, 3, 5 and 2, 4, 6, the same is not true for the lower extremes. Similar plots, with different numeric values, can be derived for the other parameters.

4.1 The Fuzzy Option Price and its Membership Function

According to the selected shape for the fuzzy parameters we compute the fuzzy prices for European options by applying the extension principle to the option pricing formula as illustrated in Section 3. We illustrate the results by considering three different strikes, which are taken as examples of out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM) options, and for two different maturities (1 and

Figure 3 Fuzzy Shapes of Parameter $\kappa*$ According to Several Definitions of the Fuzzy Support

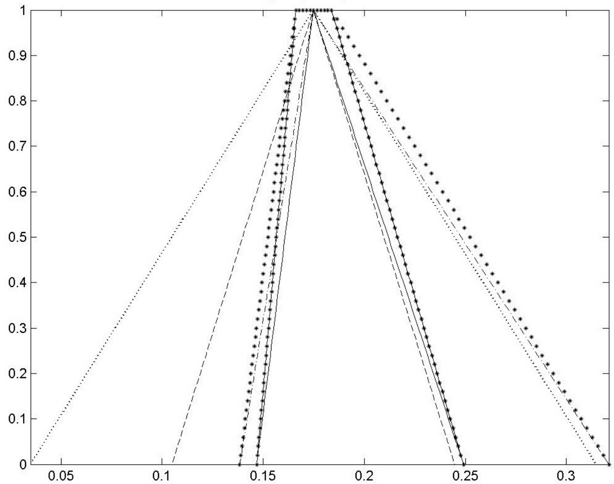
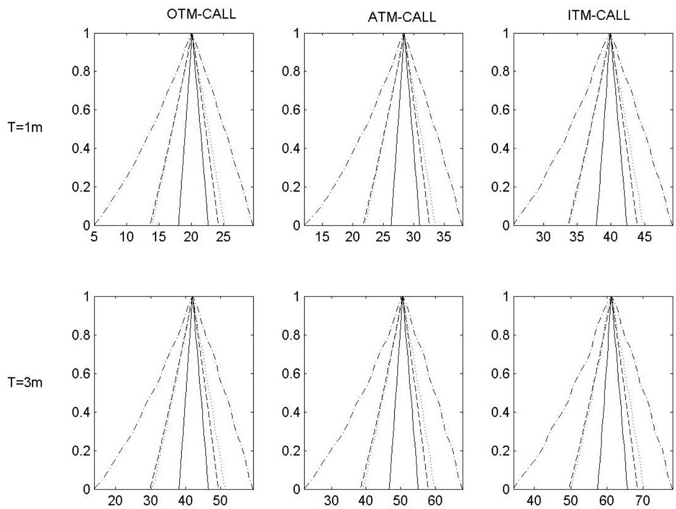
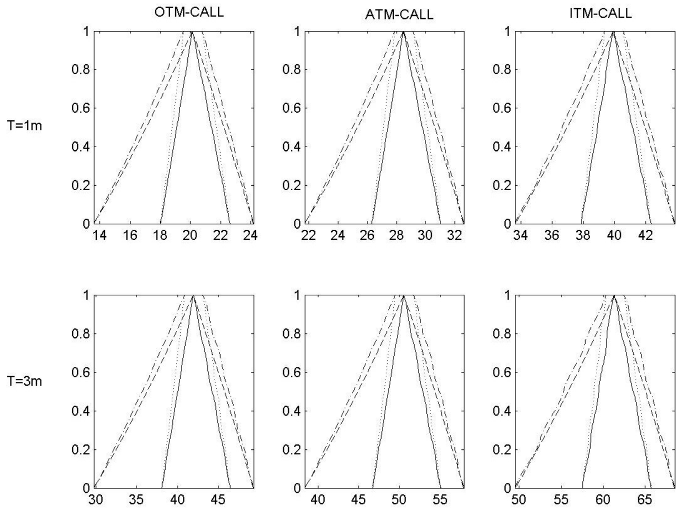


Figure 4 Fuzzy Price for Options on the S&P500 Index Traded on October, 22th, 2010: Symmetric versus Asymmetric Triangular Fuzzy Parameters



3 months). In *Figure 4*, from left to right, the membership for fuzzy option prices is reported for the examples of OTM, ATM, and ITM options, respectively; the top graphs refer to maturity T = 1 month and the bottom graphs to T = 3 months. In particular, option prices are computed starting from symmetric triangular fuzzy parameters with $p = 0.8$ (dash-dotted) and $p = 0.4$ (dotted) as well as for asymmetric triangular fuzzy parameters with extremes fixed at the 10th and 90th percentiles (dashed) and 25th and 75th percentiles (solid).

Figure 5 Fuzzy Price for Options on the S&P500 Index Traded on October, 22th, 2010: Triangular versus Trapezoidal Fuzzy Parameters



In *Figure 5*, from left to right, fuzzy option prices are reported for the examples of OTM, ATM, and ITM options, respectively; the top graphs refer to maturity T = 1 month and the bottom graphs to T = 3 months. Option prices are computed in this case assuming asymmetric triangular fuzzy parameters with extremes fixed at the 10th and 90th percentiles (dashed) and 25th and 75th percentiles (solid) as well as their trapezoidal generalization considering the 45th and the 55th percentiles as extremes for the 1-cut of the parameters' fuzzy shape (dash-dotted and dotted, respectively).

From *Figure 4* we understand how the call prices vary with the choice of symmetric or asymmetric triangular fuzzy parameter (Cases 1 and 2 against Cases 3 and 4), while *Figure 5* shows the variation of the call prices with the choice of triangular or trapezoidal asymmetric fuzzy parameter (Cases 3 and 4 against Cases 5 and 6). It is worth noting that starting from linear or symmetric fuzzy parameters does not give linear or symmetric option prices; this is due, of course, to the high non-linearity of the Heston option pricing formula with respect to the underlying model parameters.

To get a better idea of how option prices vary against the strike price, we plot in *Figures 6* to *8* the membership (shape) functions for the considered call option prices against all the available strike prices (denoted by K) on the date of interest (October 22, 2010), for maturities T = 1, 2, 3 months, respectively. The plots correspond to a choice of asymmetric triangular fuzzy parameters with the 10th and 90th percentiles as the extremes of their support (Case 3).

It is clear from the figures that not only do prices increase with the time to maturity for each fixed strike, but also that the membership *α-cuts* for fixed levels α of possibility, become larger and larger with the time to maturity of the options. In addition, the *α-cuts* ranges increase with the strike price and are more asym-

Figure 6 The Fuzzy Smile for Options with Maturity *T* = 1 Month
Fuzzy Memberships for the Option Prices Is Plotted Against the Strikes Prices Traded on October, 22th, 2010

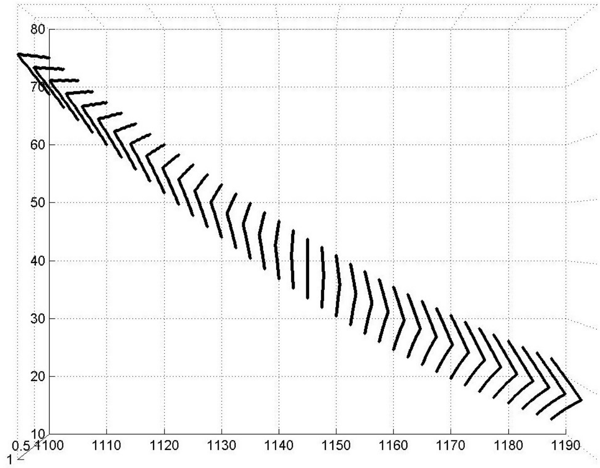
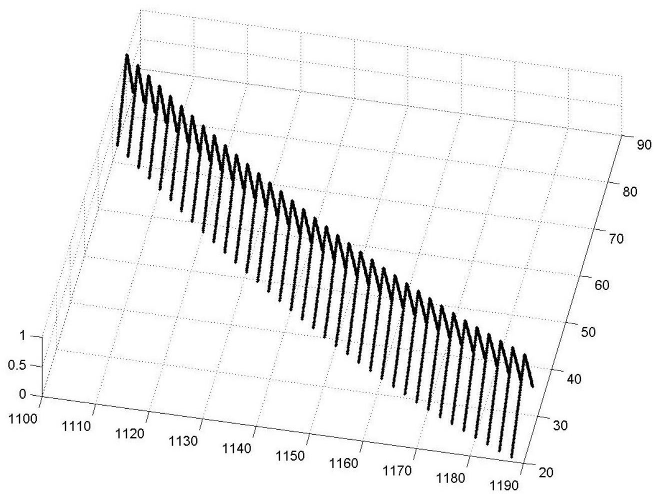


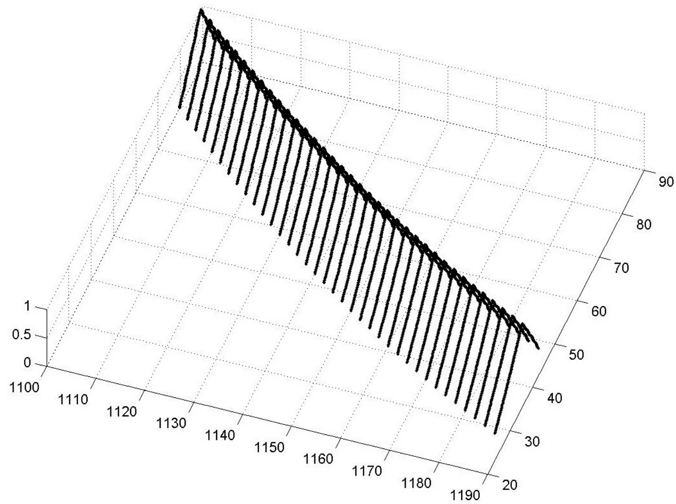
Figure 7 The Fuzzy Smile for Options with Maturity *T* = 2 Month
Fuzzy Memberships for the Option Prices Is Plotted Against the Strikes Prices Traded on October, 22th, 2010



metric for lower strike prices. Similar graphs can be obtained for the other shapes of the fuzzy parameters, leading to analogous comments.

In all the above figures the maximum possibility value $(\alpha = 1)$ corresponds to the crisp option price under the Heston model with crisp parameters. Note that in the derivative market, traded options are not quoted with a single price but with a bid price and an ask price. The difference between these prices is called the bid-ask spread

Figure 8 The Fuzzy Smile for Options with Maturity T = 3 Month Fuzzy Memberships for the Option Prices Is Plotted Against the Strikes Prices Traded on October, 22th, 2010



and, in standard option pricing models, is considered a friction in the market. In fact, a standard assumption in pricing models is that the price of a financial asset is unique; this problem is usually overcome by considering the midpoint of the bid-ask range as the unique price. In our generalized Heston framework, with fuzzy parameters, we are given a range for option prices (for each *α-cut* with $\alpha \neq 1$) which may properly represent the bid-ask spread.

5. Conclusion

Our analysis is intended to cope jointly with the uncertainty arising from the random evolution of a stock price and the vagueness of information on the assigned parameters. This fuzzy-stochastic approach is developed here within the framework of the Heston stochastic volatility model by formally representing the vagueness of the parameters through the mathematical theory of fuzzy numbers.

The Heston model provides a closed formula for the price of European options when the parameters are known. By assigning different membership functions to the fuzzy parameters we obtain, by rigorous application of the extension principle, the membership function of the fuzzy option value. It is worth noting that, due to the high non-linearity of the Heston option pricing formula with respect to the parameters, linear shapes for the membership function of the input parameters propagate to non-linear shapes for fuzzy option prices. This observation motivates the parametric representation of the fuzzy numbers involved.

In the central part of the paper we suggest a constructive method for determining the shape of the fuzzy parameters so that their memberships are properly assigned to be consistent with the empirical observations in the stock and derivatives markets.

Our analysis is developed through several experiments in which a different shape for the fuzzy parameters is assigned. We then give a numerical illustration on a dataset for the S&P 500 index and the VIX volatility index. The outcomes of our study seem encouraging and might prove useful in real markets, especially where transaction costs, such as different bid-ask prices, are indeed allowed; in this case there is no unique price for a traded asset, but instead there is a bid price and an ask price, whose range is called the bid-ask spread. Under suitable assumptions on the shape and the support of the fuzzy parameters, the extremes of the support (or of a specific *α-cut*) of the fuzzy option price, obtained using our approach, may be used to represent the bid-ask spread for the option itself. The next step for our research is to understand the statistical theoretical properties of the joint fuzzy-stochastic Heston model and possibly to apply our approach for risk measurement purposes.

As a by-product of this study we introduce a way to derive parameters in the Heston model by using a moment-matching estimation procedure based on joint observations of a stock market index and its volatility index, which served as a proxy for the integrated volatility. The proposed estimation method, though novel and very simple, is not the main focus of the paper. Of course, we will investigate this method further in future research by deriving its theoretical properties and by comparing its performance with that of existing procedures.

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