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Model Dependency of the Digital Option Replication
Replication under an Incomplete Model

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1. Introduction

A digital option is a special type of financial derivative with a non-linear discontinuous payoff function. In spite of this, the payoff is simple enough to allow (relatively) easy valuation of these contracts. It is the reason why digital options can be (and regularly are) applied to decompose and hedge statically the positions in many options with more complicated and usually discontinuous payoff functions; see for example (Andersen – Andreasen – Elizier, 2002), (Carr – Chou, 1997), (Carr – Ellis – Gupta, 1998) or (Derman – Ergener – Kani, 1995).

It is clear, that in order to ensure efficient risk management of complicated exotic options, the procedures for pricing and hedging digital options must be no less efficient. However, the valuation of digital options is easy only in the Black and Scholes (1973) setting. By contrast, relaxing some Black and Scholes restrictions can cause an incompleteness of the model, either by stochastic volatility, presence of jumps or non-normally distributed returns (non-normality of returns is commonly modeled by suitable Lévy models with an infinite intensity of jumps).

Another problem arises if a trader is not sure about the underlying model. Hence, he or she can only guess and, therefore, probably applies the incorrect one. Obviously, it can also happen that the only model available to apply is the Black and Scholes model. Although the trader can know the true evolution, it can be comprised of such complicated features that the application can be impossible. This situation can again lead to incorrect results.

Several authors have already been concerned with the efficiency of replication methods under misspecified input data. The majority of them studied the case of wrong volatility, see e.g. (Ahn et al., 1996). By contrast, in many works on static replication it was argued that to get perfect results, the underlying process can be regarded as irrelevant only if the market prices all assets efficiently.

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Another issue is how the replication works if some inputs are well-specified; however, others are not included in the model. The task of the paper is to examine the method of dynamic replication in such a case via Monte Carlo simulation and compare the results with the static replication. We also verify the irrelevance of the underlying process on the effect of static replication in the case of market efficiency.

We study two basic approaches to option replication (static and dynamic) within four different settings or different types of scenarios of the underlying asset price evolution. The only same things valid under each model are that the trader is not able to trade continuously and the only applicable model of the underlying asset price to execute the replication strategy is the Black and Scholes model. By contrast, we suppose that the market works efficiently so that the market prices are in accordance with the "true" asset price evolution.

The considered evolutions of the option underlying asset price are (i) geometric Brownian motion (thus the Black and Scholes model, BS model), (ii) geometric Brownian motion with stochastic volatility following the Hull and White process (Hull – White, 1987) (thus the stochastic volatility model, SV model), (iii) variance gamma process (Madan – Seneta, 1990) regarded as a Brownian motion subordinated by a gamma time process (thus a special type of the exponential Lévy model, VG model), and (iv) variance gamma process with stochastic volatility (or more generally the model in stochastic environment, VGSE model) driven by additional random time – the Cox-Ingersoll-Ross process (1985). The last two examples can be regarded as special cases of subordinated or time-changed processes, developed by Bochner (1949), first introduced in economics probably by Clark (1973), and tested in econometrics, e.g. by Stock (1988).

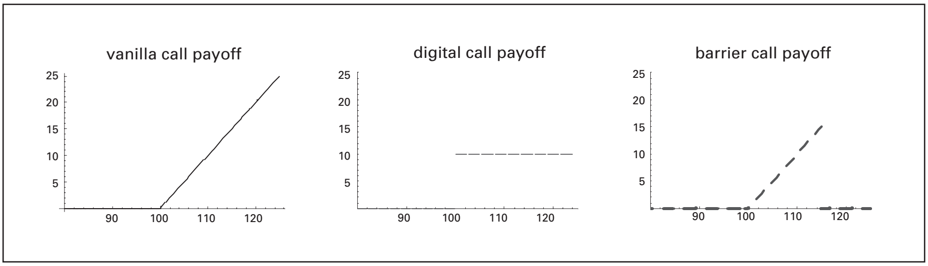
The paper proceeds as follows: In the following Section 2 we briefly review the typology of options, digital options and their pricing. Subsequently, we recall the principles of dynamic and static replication of options, see e.g. (Tichý, 2004). Next, we define and briefly describe all stochastic processes applied in this paper.[1] Finally, in Section 5 we provide numerical results of the replication performance. The paper ends with conclusions.

2. Digital Option

By an option we generally mean a non-linear financial derivative that gives its owner (*long position*) the right to buy (*call* options) or the right to sell (*put* options) the underlying asset (S) under predefined conditions. Simultaneously, the seller of the option has an obligation to respect the right of the owner (hence the *short position*). The predefined conditions concern, for example, the underlying amount of assets, the maturity time (T), the exercise price (K). Options, whose payoff can be written as $\Psi_T = (S_T - K)^+$ or $\Psi_T = (K - S_T)^+$, where S_T indicates the underlying asset price at maturity and $X^+ \equiv \max(X;0)$, are referred to as plain vanilla options.

[1] Basic reference for options, hedging and stochastic processes is (Hull, 2005). More advanced texts are e.g. (Mussiela – Rutkowski, 2004) or (Cont – Tankov, 2004).

FIGURE 1 Option Payoff Function



On the left – plain vanilla call; in the middle – digital call (Q = 10); on the right – up-and-out call (*barrier* = 115). Vertical axis – option payoff, horizontal axis – underlying asset price, exercise price K = 100.

Sometimes also other non-standard conditions are defined (average price, barrier level, etc.) and such options are referred to as exotic options. Moreover, the option which one can exercise only on the maturity date is called a European option and the option exercisable at any time at or prior to maturity is called an American option.

In this paper we focus on replication of European digital options. A digital option is a financial derivative which if exercised pays its owner some fixed amount Q or the value of any specified asset. Hence, unlike the plain vanilla call or put, the payoff does not depend on the difference between the spot price and the exercise price. It indicates that the payoff function is not smooth. Obviously, the payoff conditions of digital options can be further complicated, e.g. by the existence of a barrier level or a gap in the payoff.

The simplest example of a digital option is the *cash-or-nothing call* (*put*). The owner of this option receives at option maturity T the specified amount Q if the terminal price of the underlying asset S_T is (not) at least as high as the exercise price K. The payoff function of the call $\Psi_{call}^{dig-cash}$ can be written as:

$$\Psi_{call}^{dig-cash}(S_T, K;Q) = \begin{cases} Q & \text{if } S_T \geq K \\ 0 & \text{otherwise (if } S_T < K) \end{cases} \tag{1}$$

The situation is illustrated in *Figure 1*. The payoff of the plain vanilla call is strictly and linearly increasing for all $S > K$. The payoff of the digital call with fixed payable amount Q is horizontal with a discontinuity (and a jump) at $S = K$. Notice that the payoff of discontinuous path-dependent options is further complicated. For example, for the case of an up-and-out call it is zero up to the exercise price level, then it is linearly increasing up to $S <$ *barrier*, and finally, it is zero for all $S >$ *barrier*.

The pricing of digital options as the one of (1) is significantly dependent on the pricing of plain vanilla options. As we can see at the maturity time (see Figure 1), the value of option f_T is either zero or Q times one. Consider $Q = 1$. Apparently, the option value at maturity is identical with the vanilla option delta (the first partial derivative with respect to the underlying asset price $\Delta = \frac{\partial f}{\partial S}$). Before the maturity time, the crucial variable to value the digital option will be the probability that $S > K$.[2] In other words, it is the probability that the call option will be exercised.

From the Black and Scholes model for the vanilla call on a non-dividend stock we know that the option delta is very close to the probability of exercising, since:

$$V_{call}^{vanilla} = S \cdot N(d_+) - K \cdot e^{-r \cdot \tau} \cdot N(d_-) \tag{2}$$

where τ is the time to maturity, $N(z)$ is the cumulative distribution function of the standard normal distribution – the probability that a random number from the standard normal distribution $N[0;1]$ will be lower than z, and the term $N(d_-)$ can be interpreted as the probability that $S_T \geq K$. By contrast, the term $N(d_+)$ is the delta of the option and (within the BS model) $d_+ = d_- + \sigma \cdot \sqrt{\tau}$.

Thus, the properties of the digital call value function are close to the delta function of the vanilla call. Similarly, the digital call delta will be close to the gamma of the vanilla call.

To summarize, the digital cash-or-nothing call option value is the present value of the payoff amount Q times the probability of exercising the option:

$$V_{call}^{digital} = Q \cdot e^{-r \cdot \tau} \cdot N(d_-) \tag{3}$$

This result is general enough to be valid for various types of underlying processes. Still, the digital option price will be given by the probability of exercising, which should be very close to the vanilla call delta.

3. Digital Option Replication

Respecting the number of revisions in time we can distinguish *dynamic replication* and *static replication*. The main drawback of *dynamic replication* is implied by its definition – the method is based on an ever-changing replicating portfolio which consists of one riskless (riskless zero-bond or bank account B) and n risky assets where n is the number of underlying (independent) risk factors. Considering the Black and Scholes model, it is the underlying asset S, so that $n = 1$, and the replicating portfolio can be denoted as $H(B,S)$.

The portfolio composition can be described at general time t as:

$$H_t = x_t \cdot B_t + \Delta_t \cdot S_t \tag{4}$$

Here $(x_t;\Delta_t)$ indicates the structure of the replicating portfolio H at time t, more particularly the capital invested into the risky asset S and riskless asset B at time t.

Since the only source of risk is the underlying asset price and Δ denotes the sensitivity of the option price f to the underlying price, the risk of both

[2] Note that since the continuous-time process is supposed, the probability $\Pr(S_T \geq K)$ should be equal to $\Pr(S_T > K)$.

positions, the derivative f and the portfolio H should be the same. If that is the case and $f_t = H_t$, then it must also hold that at time $t + dt$:

$$H_{t+dt} = x_t \cdot B_{t+dt} + \Delta_t \cdot S_{t+dt} \wedge f_{t+dt} = H_{t+dt} \tag{5}$$

This furthermore implies the self-financing condition:

$$H_{t+dt} = x_t \cdot B_{t+dt} + \Delta_t \cdot S_{t+dt} = x_{t+dt} \cdot B_{t+dt} + \Delta_{t+dt} \cdot S_{t+dt} \tag{6}$$

It is clear that the weights of assets B and S in the portfolio H must be rebalanced at any time the underlying asset price changes. Hence, it needs to be done continuously, which is obviously impossible. Even if the rebalancing interval is small, but not infinitely, the replication error can be high through the discontinuity in the payoff.

Static option replication is based on static decomposition of complicated payoffs into a set of more simple payoffs. The decomposition should be such that it will be sufficient to leave the structure of the portfolio intact up to maturity. Fortunately, the payoff of a cash-or-nothing call can be easily decomposed into a tight spread of plain vanilla options.

Consider a digital call option $f = V_{call}^{dig-cash}(\tau; S, K; Q)$, where τ is the time to maturity, S is the underlying asset, K is the exercise price and the payoff amount is $Q = 1$. Create a portfolio $H_{[K-\alpha; -K]}^{vanilla\ call}$ of vanilla call options with the same time to maturity τ and written on the same asset S as indicated below:

$$H_{[K-\alpha; -K]}^{vanilla\ call} = V_{call}^{vanilla}(\tau; S, K-\alpha) - V_{call}^{vanilla}(\tau; S, K) \tag{7}$$

Hence, it is the long position in the call with exercise price equal to $K - \alpha$, $\alpha > 0$, and the short position in the call with exercise price K. At maturity the payoff will be:

$$H_{[K-\alpha; -K]}^{vanilla\ call} = (S_T - (K - \alpha))^+ - (S_T - K)^+ = \begin{cases} S_T \geq K & S_T - K + \alpha - S_T + K = \alpha \\ S_T \in (K-\alpha; K) & 0 \leq S_T - K + \alpha \leq \alpha \\ S_T \leq K-\alpha & 0 \end{cases} \tag{8}$$

Comparing equations (1) and (8) we can see that if the portfolio payoff is zero, the digital also pays zero. If the portfolio payoff is α, the digital pays Q. Therefore, by creating $x = Q/\alpha$ tight spreads, the digital payoff will be replicated. Moreover, we can see that if $S_T \in (K-\alpha; K)$ the portfolio's payoff is between zero and α. Since the digital option is not exercised for $S_T \in$ $\in (K-\alpha; K)$, the position in x portfolios $H_{[K-\alpha; -K]}^{vanilla\ call}$ is super-replicating – its terminal value will be at least the same as the terminal value of the digital option with probability one:

$$\Pr(H_T \geq f_T) = 1 \tag{9}$$

It is clear that the initial value of the portfolio cannot be lower than the derivative price under no-arbitrage condition, since it is super-replicating. Note that the portfolio $H_{[K-\alpha; -K]}^{vanilla\ call}$ will be almost replicating

$$\Pr(H_T = f_T) \to 1$$

if $\alpha \to 0$ and thus $x \to \infty$.

Alternatively, we can create a lower-cost sub-replicating portfolio $H^{vanilla\ call}_{[K;\,-\,(K+\alpha)]}$ and $\Pr(H_T \leq f_T) = 1$.

4. Stochastic Processes

In this section we briefly define all processes applied in the paper. The simplest building blocks are the Poisson process (or closely related ones such as a gamma process) and the Wiener process, which provides ingredients for construction of almost all processes with a diffusion part.

The Wiener process w_{dt} can be defined as $w_{dt} = \tilde{\varepsilon}_1 \cdot \sqrt{dt}$, where random number $\tilde{\varepsilon}_1$ belongs to the standard normal distribution, thus $\tilde{\varepsilon}_1 \in N[0;1]$, and dt describes the (infinitesimal) time increment. Hence, the Wiener process is a martingale; its expected increment is zero at any time and the variance is closely related to the time change.

We can, besides others, build on the basis of the Wiener process the *geometric Brownian motion* (GBM). It is the process which was supposed to be the one followed by stock prices in Black and Scholes (1973).[3] The typical property is the normal distribution of asset returns and logarithms of prices – which is equivalent to lognormal distribution of prices. Two key facts are that the financial-assets gain return continuously and that their prices cannot be negative. Both ideas are supported by GBM, since the price is given by an exponential formula.

It is assumed that the price dynamic can be described by the following stochastic differential equation

$$dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot w_{dt} \tag{10}$$

where dS is the price change over time interval dt, μ is the (continuous-time) expected return and σ is its volatility, both μ and σ are supposed to be deterministic constants. The solution to stochastic differential equation (10) is according to Itô's lemma:

$$S_{t+dt} = S_t \,.\, \exp\left[\left(\mu - \tfrac{\sigma^2}{2}\right) .\, dt + \sigma \,.\, w_{dt}\right] \tag{11}$$

Note also that in the risk-neutral setting the preceding formulation changes by $\mu \to r$ to ensure that the asset gains riskless return r.

Since the volatility of asset returns is very difficult to measure and forecast, some slightly more realistic models suppose its stochastic feature. However, a candidate to model the volatility must respect the empirical fact that it regularly reverts back to its long-run equilibrium. Besides others, it is the case of the *Hull and White* (HW) *model* (1987). Hull and White supposed the volatility to put into (11) can be modeled by:

[3] Although it is known at least starting from Fama's work (1965) that the financial returns are not-normally distributed, the geometric Brownian motion has been the most commonly applied process to model asset prices and to price financial derivatives.

$$d\sigma = a \cdot \sigma \cdot (b - \sigma) \cdot dt + s \cdot \sigma \cdot w_{dt} \tag{12}$$

Here, a describes the tendency of mean-reversion, b is the long-run mean (equilibrium) and s is the volatility of the volatility. The Wiener process of HW (12) which drives the volatility is usually supposed to be independent of the one of the GBM (10).

4.1 Lévy Models

Under the family of Lévy processes, so called in honor of Paul Lévy, are generally understood such processes that are of independent and stationary increments. These processes are also typical by the stochastic continuity – the probability of jump occurrence for given time t is zero. The Lévy process can be decomposed into a diffusion part and a jump part. Clearly, not all parts must be present.

The modeling of financial prices is usually restricted to exponential Lévy models. The price dynamic is given by an exponential of a Lévy process X_t and some (deterministic) drift μ:

$$S_{t+dt} = S_t \, . \, \exp[\mu \, . \, dt + X_{dt} - \varpi \, . \, dt] \tag{13}$$

Moreover, we must deduce the term $\exp(\varpi) = \exp(E[X_{dt}])$ to ensure that $E[S_{t+dt}] = S_t \, . \, e^{\mu \cdot dt}$. In fact, it is equivalent to deducing $\frac{1}{2} \, . \, \sigma^2 \, . \, dt$ in the case of geometric Brownian motion. We can therefore interpret ϖ as a mean correcting parameter to the exponential of the Lévy process X_t.

The classic works incorporating jumps in price returns were based on jump-diffusion models such as the Merton model (1976). These models are typical by a finite number of jumps in any time interval. However, the modern models of financial returns are of infinite activity – thus, the jumps, although small in scale, occur infinitely many times in any time interval. In fact these models do not need to be constructed of diffusion components, since the infinite activity allows description of the true feature (either jumps or skewness and kurtosis in the distribution of returns) well enough. In addition, the terminal price can be produced by simulation within one step.

Many Lévy models are regarded as subordinated Brownian motions. If $w(t)$ denotes a Wiener process in time t, we can define the subordinated Brownian motion X_t with drift μ and volatility σ by subordinating with another Lévy process $g(t)$ just replacing t by $g(t)$. Thus, changing the time we get:

$$X_t = \mu \cdot g(t) + \sigma \cdot w(g(t)) \tag{14}$$

Hence, the subordinated process $X_t(g(t);\mu,\sigma)$ is driven by another process $g(t)$ which is referred to as the subordinator. Since the random process $g(t)$ plays the role of the time in the original model, it must be non-decreasing in time.

In such a case we have to imagine "an internal time" given by process $g(t)$. Of course, the process still evolves in time t. However, so-called internal time gives us a very nice economic interpretation of subordinated processes

– the (geometric) Brownian motion given in a random business time, which is stipulated by the economic activity, the mass of information, etc.[4] In other words, "the time increments" are not constant but stochastic. Transforming the original time into the stochastic process we can also model other parameters of the distribution and fit the model more closely to the set of real data.

The very popular subordinators are the *gamma process* resulting into *variance gamma model* (the name is due to the variance of the primary component is not given by the classical time but by the "gamma-time") and the *inverse Gaussian process* which results into *normal inverse Gaussian model* – see e.g. (Barndorff-Nielsen, 1995).

In this paper we apply the variance gamma (VG) model[5] (for more details see, e.g., (Madan – Seneta, 1990), (Madan – Milne, 1991) or (Madan – Carr – Chang, 1998)). Consider the VG process $VG(g(t;\nu);\theta,\vartheta)$, where $g(t;\nu)$ is the (random, but strictly increasing) gamma time from gamma distribution $G[1;\nu]$ (here ν describes its variance and allows us to control the kurtosis), θ is the drift (by which we can control the symmetry), and ϑ describes the volatility. Hence the asset price dynamics can be expressed as:[6]

$$S_{t+dt} = S_t \cdot \exp[\mu \cdot t + VG_t - \varpi \cdot dt] = S_t \cdot \exp[\mu \cdot t + \theta \cdot g_t + \vartheta \cdot w(g_t) - \varpi \cdot dt] \tag{15}$$

where $\varpi = -\frac{1}{2} \cdot \ln(1 - \theta \cdot \nu - \frac{1}{2} \cdot \vartheta^2)$.

One further step is to incorporate the notion of the stochastic environment into Lévy models. Although many Lévy models allow fitting well the empirical structure of returns including skewness and kurtosis, the calibrated parameters in general do not stay the same over time. Besides the stochastic volatility approach of Hull and White (1987) or Heston (1993), this can be done either by applying Lévy-driven Ornstein-Uhlenbeck processes to model volatility (this direction was developed mainly by Barndorff--Nielsen and Shephard) or time changed Lévy processes (which was suggested by Carr et al. (2003)). A brief review of all approaches are provided by, e.g. (Cont – Tankov, 2004) or (Schoutens, 2003).

Here, we proceed according to the approach of Carr et al. (2003), in which according to Brownian scaling property it is supposed that the change in volatility can be captured by the (random) change in time. Thus, although the VG model is given by time-changed Brownian motion (by gamma time), it is further extended by introducing a stochastic time $Y(t)$ given by mean--reverting CIR square-root process (Cox – Ingersoll – Ross, 1985):

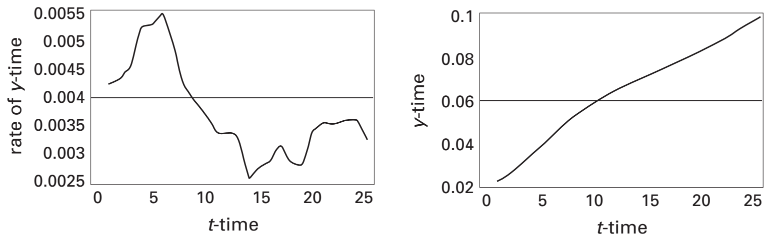
$$dy = \kappa \,.\, (\eta - y)\, dt + \lambda \,.\, \sqrt{y} \,.\, w_{dt} \tag{16}$$

[4] For example, if the economic activity is above average, the internal time grows rapidly comparing with the classical time. And *vice versa*.

[5] VG model can be alternatively defined as a difference between two increasing gamma process, one for positive increments in the price, the other for negative ones.

[6] Note, that this is the true (statistical) evolution of the price. However, to price a financial derivative we need to change the drift to be risk-neutral and, probably, also change other parameters of the VG process.

FIGURE 2 Illustration of the Time-Changed Process – CIR



with long-run time change η, the rate of mean reversion κ and time volatility λ. Thus, the $VG(g(t;\nu);\theta,\vartheta)$ model can be reformulated into $VG(g(t;\kappa,\lambda);\ \nu);\ \theta,\vartheta)$. Note that (16) describes the dynamics of the *time rate y* – the change of Y-time over the interval dt. Thus, $y_{t+dt} = y_t + dy$ and the alternate time describing the stochastic environment is given by:

$$Y_t = \int_0^t y_u \mathrm{d}u \tag{17}$$

Figure 2 illustrates the evolution of these two times, t and y, for $\Delta t = 0.004$. The rate of this internal time y is illustrated on the left and the accumulated time Y is on the right.

As before, to get the asset price dynamic in either a true or risk-neutral setting, we must incorporate the mean correcting parameter. For example, in the risk-neutral setting we need to get $E[S_{t+dt}] = S_t \ .\ e^{r \cdot dt}$. And therefore

$$S_{t+dt} = S_t \ \frac{\exp[r\ .\ dt + VG(Y(dt))]}{E[\exp[VG(Y(dt))]]} \tag{18}$$

Fortunately, the replication should be valid regardless of the type of the world. Hence, in order to examine its efficiency we can stay within the risky (or true) market feature.

4.2 Option Pricing within Lévy Models

Lévy models must usually be regarded as incomplete. Standard Black and Scholes arguments (replication with the underlying) cannot be used since there are more sources of risk. The alternative risk-neutral approach is also problematic since there does not exist a unique martingale probability which is equivalent to the original space of true-market probabilities. The pricing problem can be solved by incorporating a mean correcting parameter, introducing characteristics functions or applying suitable transform techniques. Some interesting questions of martingale measures of Lévy processes are examined by, e.g., Fujiwara and Miyahara (2003).

Note that the nature of Lévy processes usually does not allow us to use dynamic replication of Section 3 as it is valid only for the world of one relevant risk factor. Moreover, it is often very difficult to replicate the risk of jumps.

For illustrative reasons, we will now state the European call option pricing formula within VG model $V^{VG}(S,\vartheta;\tau)$, which is probably the only one available in the "user-friendly" expression:

$$V^{VG}(S,\vartheta;\tau) = \int_0^\infty G(t)\, V^{BS}\left(S \cdot \exp(\theta g + \tfrac{1}{2}\vartheta^2 g - \omega\tau),\ \vartheta\sqrt{\tfrac{g}{\tau}};\tau\right) \mathrm{d}g \qquad (19)$$

As before, S is the underlying asset price, ϑ is the volatility, θ is the drift, τ is the time to maturity, ω is the mean correcting parameter, $V^{BS}(\cdot)$ is the Black and Scholes pricing formula and $G(t)$ denotes the probability density function of the gamma distribution.

5. Numerical Results

In this section we successively apply particular models in order to verify the efficiency and examine the differences of dynamic and static replication under various (in)complete models. This is done by simulating 10,000 random paths of the underlying asset price evolution and calculating the characteristics of the terminal replication error $(H_T - f_T)$.

We suppose four different types of underlying models: BS (Black and Scholes) model, SV (stochastic volatility Hull and White) model, VG (variance gamma) model and VGSE (VG in stochastic environment given by CIR model) model.

Furthermore, we suppose that the market works efficiently and the only model ready to be applied when executing the replication/hedging strategy is the BS model. The reason can be either unknowingness or ignorance of the true model or the impossibility of its application – although the true model of the evolution can be known, it need not be feasible to work with it. As the number of independent factors and complexity of the model increases, it starts to be more and more time-consuming to get the option fair price. Calculation of optimal portfolio positions is further complicated. Normally, some numerical approximation technique should be used.

In each of the tables 1–4 we provide the characteristics of replication error distribution. In particular, the first part includes the minimal (*min*) and maximal (*max*) error value. The second part describes the basic characteristics of the distribution: mean, median and the standard deviation (σ). The third part gives more information about the shape of the distribution – it provides the skewness and kurtosis.

Consider now a financial institution whose task is to replicate a written option f as efficiently as possible. The option parameters are:

$$f = V_{call}^{dig/cash}\left(\tau = 0.1;\ S_0 = K = 100;\ \mu = r = 0.1;\ \sigma = 0.15;\ Q = 1\right)$$

FIGURE 3 Dynamic Replication Error of Digital ATM Call – BS/BS, M = 5, 25, 100, 600

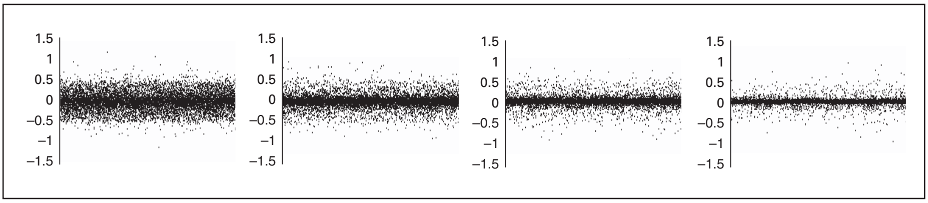


TABLE 1 Dynamic Replication Error of Digital ATM Call – BS/BS

M	Min.	Max.	Mean	Median	σ	Skewness	Kurtosis
weekly	–1.187	1.255	–0.004	–0.003	0.243	0.040	3.408
daily	–0.949	1.008	–0.001	–0.001	0.173	0.056	6.739
6 hours	–0.996	1.141	0.001	0.000	0.126	–0.090	14.296
1 hour	–1.039	0.985	0.000	0.000	0.080	–0.179	35.708

5.1 Dynamic Replication

In this subsection we examine dynamic replication of the option f by (discrete) trading with the underlying S and the risky asset B.

Firstly, we examine the complete BS model. Thus the price of the underlying asset S follows the SDE (10) so that the option is priced and can be hedged due to the BS model. The initial value of the option equals the initial value of the replicating portfolio $H_0 = f_0 = 0.5685$ and the initial position in the risky underlying asset is given by $\Delta_0 = 0.0818$. The replication error at maturity time is given by $RE_T = -f_T + H_T$. The asset price evolution during option life is recovered according to the discrete version of (11):

$$S^{(n)}_{t+\Delta t} = S^{(n)}_t \,.\, \exp(\Delta S^{(n)}_{\Delta t}) \; = \; S_t \,.\, \exp[(\mu - \tfrac{\sigma^2}{2}) \,.\, \Delta t \; + \; \sigma \,.\, \widetilde{\varepsilon}^{(n)}_1 \,.\, \sqrt{\Delta t}] \qquad n=1,\ldots,N$$

Here, N = 10,000 is the number of independent paths and Δt is obtained according to pre-specified number of discrete rebalancing intervals, $M = {}^{\tau}/_{\Delta t}$. Since the option life supposed here is 5 weeks, which is about 25 business days, the considered rebalancing intervals are once a week, one day, six hours and one hour, i.e. M = 5, 25, 100, 600.

The resulting values for particular rebalancing intervals are presented in *Figure 3* and *Table 1*. The graphical results indicate the symmetry distribution of the error around the zero-mean and decreasing standard deviation with an increasing number of rebalancing intervals. However, some extreme results are present even for very short intervals.

All these results are confirmed by exact values of particular characteristics in Table 1. Compare daily and hour rebalancing. Although the mean is zero and the standard deviation decreases significantly, the extreme results – minimal and maximal errors – are almost the same (or even slightly higher). Note that the maximal payoff of the option is one. It means that the maximal shortfall is higher than the maximal payoff.

FIGURE 4 Dynamic Replication Error of Digital ATM Call – BS + SV/BS

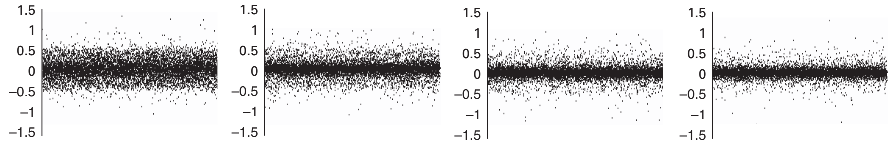


TABLE 2 Dynamic Replication Error of Digital ATM Call – BS + SV/BS

M	Min.	Max.	Mean	Median	σ	Skewness	Kurtosis
weekly	–1.15291	1.65711	–0.006	–0.006	0.258	0.124	3.769
daily	–1.55101	1.42682	–0.003	–0.004	0.203	0.164	6.975
6 hours	–1.22939	1.13906	0.000	–0.002	0.162	–0.048	10.125
1 hour	–1.338	1.337	–0.001	–0.001	0.127	0.005	13.926

The skewness and the kurtosis are closely connected with the intensity of trading (rebalancing). In particular $M = 600$ indicates obviously negative skewness and high kurtosis. The (a)symmetry is also confirmed by the quantiles given in the same table.

Secondly, we proceed to the case of stochastic volatility given by the Hull and White (*SV model*). In order to model the price we first generate the volatility according to equation (12) and subsequently put it into (11). However, we assume that the financial institution cannot take advantage of this model and it can apply only the BS model.

By contrast, suppose that the market is efficient and the option market price corresponds with the SV model, including its parameters $\sigma_0 = b =$ $= 14.5$ % p.a., the rate of mean reversion $a = 16$ % p.a. and the volatility of volatility $s = 1$. Respecting these parameters, the market price is approximately the same as the one according to the BS model (implied volatility is close to the actual). It is caused by the fact that the payoff of the digital option is influenced only by the probability of exercising and it stays approximately the same in the example studied here.

The results are depicted in *Figure 4* and *Table 2*. The graphical presentation of results indicates symmetric distribution around zero; however, it is more spread-out this time. It is confirmed by values in the table. The stochastic volatility does not cause the occurrence of arbitrage opportunity in the broad sense, as the mean is close to zero again.

By contrast, the standard deviation is higher, compared with the BS model, for high-frequency rebalancing. (For infrequent trading it is almost the same). Also interesting are the maximal and minimal error. It is significantly higher compared to the BS model and far behind the maximal payoff value. On the other hand the kurtosis is not so high (it is caused by more spread-out distribution).

As we decrease the rebalancing interval in the BS model, we are able to mimic the true evolution almost exactly for the majority of scenarios – the de-

FIGURE 5 Dynamic Replication Error of Digital ATM Call – VG/BS

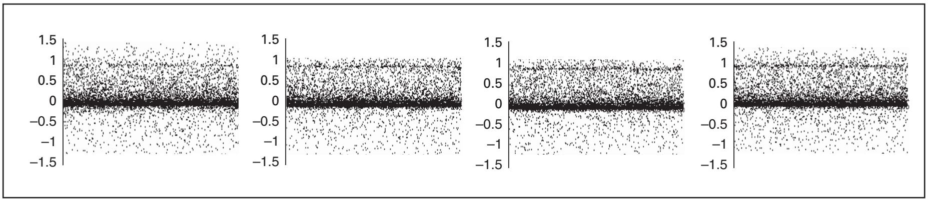


TABLE 3 Dynamic Replication Error of Digital ATM Call – VG/BS

M	Min.	Max.	Mean	Median	σ	Skewness	Kurtosis
weekly	–12.007	4.691	–0.003	–0.067	0.613	–3.312	48.296
daily	–20.601	8.832	–0.010	–0.085	0.700	–5.241	121.807
6 hours	–36.845	8.464	–0.011	–0.087	0.846	–15.400	541.493
1 hour	–13.145	10.067	0.010	–0.088	0.631	–2.944	60.355

viation decrease significantly, but the extremes stay similar. However, in the SV/BS setting, we are not able to perceive the true evolution with so much success – the true realized volatility can be different to the one we suppose.

Thirdly, we suppose that the underlying asset price returns are not normally distributed; the skewness and kurtosis are presented. Hence we model it by the VG model with the parameters $\{\theta, \vartheta, \nu\} = \{-0.1436,\ 0.12136,\ 0.3\}$, which implies volatility slightly lower than 15 %, skewness of –0.8 and kurtosis of 4.14. We suppose again that the only model available to the financial institution is the BS model.

The initial market price of the option is $f = 0.75$. The price corresponds to the VG model introduced above. The institution does not know this model, so that it calibrates the volatility to get $V_{call}^{dig/cash}(\cdot, \sigma = ?)$ The result is $\sigma = 0.045$.

As before, the results are given in *Figure 5* and *Table 3*. This time, however, the graphical presentation indicates that the results are significantly different. Although the strategy is no-arbitrage – the mean error is close to zero – the deviation is high, the extreme (minimal) results are huge, as are skewness and kurtosis. In the picture a line close to one and negative median are noticeable. We also see that frequency of trading does not play a significant role.

The bad results are caused by the fact that the VG model is typified by the occurrence of jumps (with infinite activity). These jumps are usually small, but they do exist. If it happens close to the exercise price, where the delta is very sensitive, it can cause significant errors.

Have a look at the source of the line close to 1. Remember that the terminal error has been formulated as $RE_T = -f_T + H_T$. Here, the option payoff f_T is either zero or one. Hence, the first idea is that such an error emerges if $f_T = 0$ and $H_T = 1$ – shortly before the maturity, the option is not supposed to be exercised; however, a jump (or large increment) can occur and the new conditions will cause the replication error to be close to one. The counterexample is that although the reverse can happen there is no such line close to minus one.

A more justifiable idea is that it is natural implication of skewness. Indeed, if we examine the strategy for various levels of skewness, we could see that the distribution of the terminal error is symmetric only if the underlying distribution is also symmetric. As the skewness of the underlying distribution increases (decreases), and under the no-arbitrage condition, the median becomes negative and a distinct line in the positive area appears.

The last example of dynamic strategy concerns the stochastic environment given by the time changed VG model (VGSE); see equations (15) and (16). The parameters to put into the CIR model must be such that the condition $2 \cdot \kappa \cdot \eta \geq \lambda^2$ will hold. For simplicity, we suppose that η is identical with the length of the rebalancing interval.[7] Thus the lowest η is 1/6000 and choosing $\lambda = 0.1$ we must set $\kappa = 32$ (at least). These inputs imply the market price to be close to the VG price $f = 0.75$. Because of this, the BS implied volatility is again approximately $\sigma = 0.045$.

Since we have chosen relatively low volatility and high speed of mean-reversion, the results are not very different from the VG model and therefore are not presented here. To briefly name the big difference – the deviation is somewhat lower and the mean is slightly positive (around 0.003). Instead of tables with results we have presented Figure 2 – a particular scenario hof Y-time for $\eta = 1/250$. The stochastic environment given by the time-fluctuation implies the change of all distribution: the volatility, skewness and kurtosis.

5.2 Static Replication

In this subsection we examine the static replication. Consider the same digital option $f = V_{call}^{dig/cash}(\tau = 0.1; S_0 = K = 100; \mu = r = 0.05; \sigma = 0.15; Q = 1)$ which will be statically (super)replicated by (decomposed into) the position in a vanilla call spread $H = VS = \{V_{call}^{vanilla}(0.1;100,99) - V_{call}^{vanilla}(0.1;100,100)\}$, α from equations (7) and (8) is one. The final error can arise only if $S_T \in$ $\in [99,100]$. The maximal potential error is one – S_T is close to 100. Note also that the difference between the values of these two assets H and f increases with the approaching maturity time.

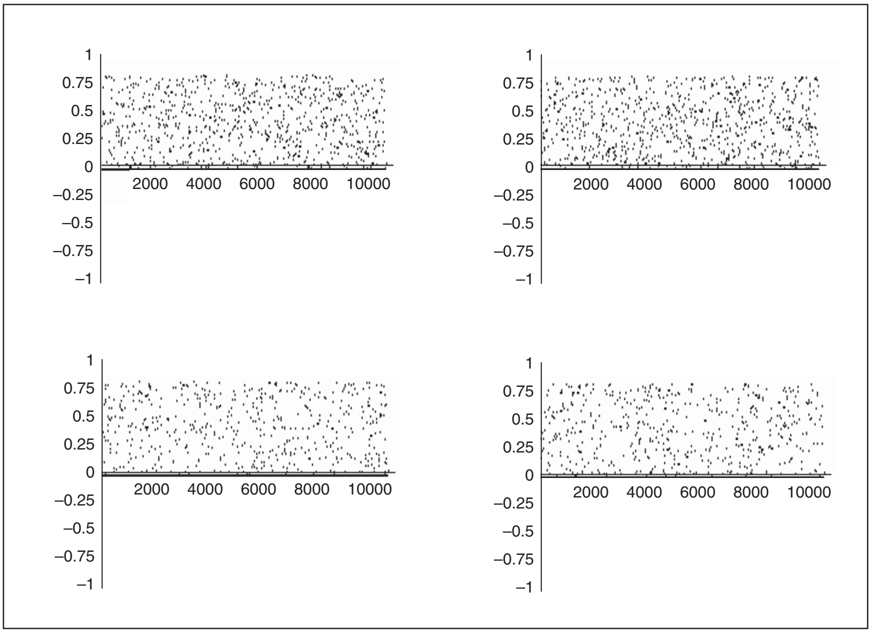
Since the portfolio is superreplicating, we must proceed as follows. At the beginning, the financial institution gets the yield from the sold option. The capital which is to be spent to long the spread is, however, higher. The difference (which is nonzero and negative) is borrowed at the riskless rate up to maturity.

In this way, we can create a portfolio Π.

$$\Pi = H_0 + (f_0 - H_0)$$

[7] Empirical estimates for the model show that the long-run rate of Y-time change need not strictly be closely related to physical time. Furthermore, some estimates may violate the non--negative condition. For more results, see (Carr et al., 2003) or (Schoutens, 2003).

FIGURE 6 Static Replication of Digital Call – BS, SV, VG and VGSE model



which should on average replicate the option, $E[f_T] = E[\Pi_T]$. All the positions are left intact up to maturity. At that time, the replication error (RE) can be formulated as:

$$RE_T = -f_T + H_T + (f_0 - H_0) \cdot \exp(r \cdot \tau) =$$
$$= -1 \cdot I_{st \geq K} + (S_T - K - 1)^+ - (S_T - K)^+ - (f_0 - H_0) \cdot \exp(r \cdot \tau)$$

First suppose the complete setting – the case of the BS model. The initial difference in values of the derivative and the replication portfolio is given by:

$$(f_0 - H_0) = (0.57 - 3.03 + 2.42) = -0.04$$

Obviously, this will determine the error if $S_T \notin [99,100]$. In particular:

$$RE_T(S_T \in [99,100]) = (f_0 - H_0) \,.\, \exp(r \,.\, \tau) \approx -0.041$$

The graphical illustration of results is given by *Figure 6* and particular characteristics are depicted in *Table 4*. Since the portfolio is left intact up to maturity, the number of rebalancing intervals is irrelevant. Therefore, only one result will be provided for each method. Note that the probability $\Pr\{S_T \in [99,100]\}$ is close to 5 %, which is approximately 500 random paths of the underlying asset price evolution.

The presentation of results confirms the theoretical bounds of the error. The total error is either –0.041 ($S_T \notin [99,100]$) or between zero and one (for

TABLE 4 Static Replication of Digital Call – BS, SV, VG and VGSE model

Model	Min.	Max.	σ	Mean	Median	Skewness	Kurtosis
BS model	–0.041	0.959	0.161	0.000	–0.041	4.212	20.275
SV model	–0.043	0.956	0.164	0.000	–0.043	4.070	19.097
VG model	–0.028	0.971	0.135	–0.001	–0.028	5.330	31.473
VGSE model	–0.030	0.967	0.136	–0.002	–0.030	5.284	30.983

$S_T \in [99,100]$, excluding initial costs). It significantly influences the median, skewness and kurtosis. The mean value of the replication error is zero, which confirms the no-arbitrage opportunity. Note again that if the mean value differs from zero, one of positions is preferred (either f or H) and the relative prices change.

The SV model, VG model and VGSE model are applied by virtue of the same principle. We suppose that actual market prices correspond to the relevant model of the underlying asset price evolution. On the basis of market prices (market prices are everything we need to know to replicate the option statically) the initial difference is calculated. It also determines the terminal error for $S_T \notin [99,100]$. The results are clear from Figure 6 and Table 4.

Apparently, these results are very similar for all models. None of the strategies allows an arbitrage opportunity; the mean is zero and the standard deviation is very low. It seems that the SV model is close to the BS model, and the VG model is close to the VGSE model, as should be supposed.

5.3 Other Market Frictions

The efficiency of both the dynamic and static replication strategies is determined also by the existence of transaction costs and liquidity or position constraints.

The dynamic strategy is based on more or less frequent trading with the underlying asset. Without doubt, as we increase the frequency of trading the total costs we incur will also rise. By contrast, the static strategy requires trading only at the beginning, when the portfolio is set up. Although the transaction costs are commonly higher on the financial derivatives market than on the spot market, the difference is usually not so high as to make the dynamic strategy more favorable. Besides that, the dynamic method of the VG/BS and VGSE/BS strategy is totally unsuitable.

Another problem arises when the market is not sufficiently liquid or constraints on portfolio positions must be respected. This issue concerns mainly trading with options.[8] As was shown in equations (7)–(9), the error of static replication can be controlled by the parameter α. As we bring the error bounds closer to zero, the number of options to be purchased and sold sharply rises. The requirement need not be met with the true market characteris-

[8] Of course, also large short positions in the underlying can be prohibited. However, the delta of digital options is low, so it should not significantly affect the dynamic replication.

tic. It can be either impossible to purchase such a high number of options or this act would be associated with unfavorable transaction costs. This was the reason why we supposed here a relatively high α, which allows us sufficient freedom in setting up the portfolio.

6. Conclusions

Digital options can be regarded as a derivative at a half-step between plain vanilla options and complicated exotic options. On one hand the (theoretical) pricing is relatively simple; on the other the discontinuity in the payoff function can cause a serious problem in replication and hedging.

The task of this paper was to examine the relationship of the replication error on the completeness of the model. More particularly, we have studied the dynamic replication and the static replication within four distinct models. Three of them were supposed to be incomplete. We showed how replication methods work if the underlying process is not known or cannot be utilized when the replication portfolio is constructed.

The dynamic method performs relatively well only in a complete setting and with frequent rebalancing of the replication portfolio. Under incomplete models, the frequency of BS-rebalancing does not play such a significant role (SV model) or it is almost insignificant (VG model).

By contrast, the static replication performed well also if the underlying process was not known by the subject. However, the inevitable assumption is that the market price must correspond to the true evolution, otherwise, the results may be poor.

The strength of the static replication is that it does not require trading up to the maturity and it allows us to manage the theoretical bounds of the replication error. Of course, if the market significantly changes its view about the underlying price process, the positions can be rebalanced to minimize the expected error.

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SUMMARY

JEL Classification: G13, G20, D52, C1
Keywords: digital options; dynamic and static replication; internal time; Lévy models; replication error; stochastic environment; stochastic volatility; variance gamma process

Model Dependency of the Digital Option Replication
Replication under an Incomplete Model

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The paper focuses on the replication of digital options under an incomplete model. Digital options are regularly applied in the hedging and static decomposition of many path-dependent options. The author examines the performance of static and dynamic replication. He considers the case of a market agent for whom the right model of the underlying asset-price evolution is not available. The observed price dynamic is supposed to follow four distinct models: (i) the Black and Scholes model, (ii) the Black and Scholes model with stochastic volatility driven by Hull and White model, (iii) the variance gamma model, defined as time changed Brownian motion, and (iv) the variance gamma model set in a stochastic environment modelled as the rate of time change via a Cox-Ingersoll-Ross model. Both static and dynamic replication methods are applied and examined within each of these settings. The author verifies the independence of the static replication on underlying processes.