UDC: 339.743;330.101.541;519.866
JEL Classification: F31, F41, G12, G15
Keywords: exchange rate; pricing kernel; order flow; latent risk; state space

Macroeconomic Factors and the Balanced Value of the Czech Koruna/Euro Exchange Rate

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1. Introduction

The present paper investigates the linkages between the key macroeconomic uncertainties present in the Czech and euro-area economies, on the one side, and key asset prices, including the Czech koruna/euro exchange rate, on the other. For this purpose, we construct a stochastic optimizing model of a small open economy. We then perform a risk-factor decomposition of the exchange rate in the said model by joining the resources of portfolio-optimization theory, factor models of asset prices and micro finance. We allow for the existence of liquidity management and other agent-specific determinants of the nominal exchange rate in a financially integrated open economy along with the purely macroeconomic fundamentals, and measure the relative importance of both. A similar model was formulated in (Derviz, 2004a).

Traditional financial economics derives restrictions on the exchange-rate dynamics in an open economy from an optimizing investor's actions (even if it does not normally pin the exchange-rate value down unambiguously). On the other hand, positivist empirical finance concentrates on decomposing the observed exchange rate into statistically well-defined components without offering much in the way of explaining their economic sources. To provide for an explanation in an environment with asset market frictions, we need a synthesis of both. The "missing link" is found in the New Micro approach (Lyons, 2001) to exchange-rate modeling. This approach constitutes a departure from the conventional representative investor (Walrasian) dynamic asset pricing paradigm in the direction of modeling elastic supply and demand environments. Following this line of thought, we model asset markets where the investor faces explicit pricing schedules, and where excess demand is absorbed by exogenous market makers.

Individual investor optimization in the chosen setting leads to a generalization of the international consumption-based asset pricing model (IC-CAPM). The well-known price formulae of the latter now turn out to be valid for a set of unobservable "shadow" prices which usually differ from

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This research was financially supported by the Grant Agency of the Czech Republic, grant No. 402/05/0671.

the actual ones, the difference determining the direction of trades. Thus, an important methodological contribution of the paper is the extension of the econometric factor asset pricing approach to this environment.

The central notion operated by our model is that of the shadow exchange-rate value. For lack of well-established alternatives, we call it the *autarchic exchange rate*. It is the value of the currency that does not induce investors to either buy or sell. In this sense, the autarchic exchange rate is the "balanced" one, although not precisely the "equilibrium" one, since the generic dynamic equilibrium that the model gives rise to implies non-zero foreign exchange (FX) transactions. That is, cross-border FX order flow is a summary statistic for all sorts of asymmetries, be it of an inventory, informational or institutional nature that may exist between resident and non-resident investors. Among other things, the model demonstrates that it is the autarchic and not the actual exchange rate for which the standard uncovered parity properties should hold.

We assume the existence of a latent risk factor, which is present in the forex order flow along with observable aggregate economic and financial uncertainties. This has consequences for asset pricing in general and for the exchange-rate behavior in particular. To be able to concentrate on the forex risks, we consider deviations of the actual from the autarchic price for the single asset (FX), assuming balanced pricing, on average, for all other assets (i.e. their autarchic and actual prices only differ by random noises). This approximation is justified by the application to be studied, since the cross-border FX order flow compared to cross-border order flow in other market segments clearly dominates in the Czech economy. That is why the model version employed by the present paper can be dubbed an *FX Order Flow Gap Model*.

The model leads us to a set of shadow asset prices that determine a particular exchange-rate value for the hypothetical no-forex trade case. This currency value is determined by the remaining prices of goods and assets in the Walrasian sense. A similar result is already implicit in the (Lucas, 1982), international consumption-based CAPM. There and in all other models built on the same foundations, the exchange rate satisfies an uncovered parity condition involving any two returns in different currencies. However, the model allows for various deviations from the standard Walrasian outcome, due to heterogeneity of endowments, information or institutional status. We capture the said heterogeneity by a summary statistic of exogenous aggregate FX order flow absorbed by an exogenous market maker. Self-fulfilling beliefs about this aggregate source of uncertainty are able to generate sunspot exchange-rate trajectories – see (Derviz, 2004b) for a model in this vein. That is, the exchange rate is co-driven by an independent source of uncertainty.

The obtained theoretical result is cast into the shape of a testable state-space model when we assume the existence of a finite number of (Gaussian) risk factors that jointly determine the dynamics of the (autarchic) asset prices and the FX order flow. These latent factors are assumed to span the uncertainties encountered by a typical investor with an open position in foreign cash. In view of the pursued objective – to detect the forex volatility sources in excess of the standard macroeconomic risks – we let the vector of state variables contain a component imperfectly correlated with the main macro-fundamentals. This residual uncertainty factor may be purely inventory-driven but one cannot *a priori* exclude that it is a reflection of structural asymmetries in the forex. The central premise of our analysis is that this extra variable shall have a significant correlation with the cross-border FX order flow.

In the empirical part, we estimate our theoretical model in state-space form on selected Czech and euro-area assets. Estimation results are reported for the autarchic CZK/EUR rate as a function of Czech macroeconomic fundamentals relative to the euro area and the EU and Czech asset returns. We select a small set of major Czech securities enlarged by a set of their counterparts in the euro area to act as observation variables. A Kalman filter procedure conducted on return processes for these assets is used to isolate the autarchic CZK/EUR exchange rate and compare it with the actual one.

The main findings of the study are as follows:

– the cross-border FX order flow, reflected in the reported Czech bank spot FX transactions with non-residents, explains a large portion of the exchange-rate movements;
– the significant risk factors influencing the observed FX order flow include both standard macro fundamentals and idiosyncratic shocks related to liquidity management;
– the detected deviations of the actual return from the autarchic return on the Czech koruna cash, as predicted by the model, point at the CZK/EUR rate bubbles associated with episodes of pronounced one-sided cross-border FX order flow.

1.1 Background and Literature Review

The distinct feature of the present model is the presence of an independent orderflow risk factor not spanned by the asset prices. Accordingly, the equilibrium can no longer be characterized in terms of risk-neutral expected values. The asset prices become individual preference-dependent. Our approach combines the standard individual optimization-based dynamic asset pricing paradigm in discrete time with several findings of micro-based FX theory. Altogether, the methodology of the paper has three main sources.

First, we draw on the literature on ***multi-factor models of asset prices and yield curves***. To explore the macroeconomic fundamentals role in asset pricing and allocation, the stochastic intertemporal optimization paradigm has been applied to both multivariate GARCH (Flavin – Wickens, 2003), (Wickens – Smith, 2002) and VAR (Ang – Piazzesi, 2003), (Bomfim, 2003) asset dynamic specifications. In both variants, one needs to create a model for the pricing kernel in terms of the relevant underlying sources of uncertainty. Sometimes, in the tradition of multifactor yield-curve modeling literature (Duffie – Kan, 1996), the underlying factors are considered totally unobservable. At the other extreme, the analysis is limited to observed macroeconomic and financial fundamentals only (Flavin – Wickens, 2003), (Wickens – Smith, 2002), (Bomfim, 2003). We employ a hybrid approach analogous to that of Ang and Piazzesi (2003), by considering a vector of observed macrofundamentals extended by a latent factor. In our model, this latent factor is directly linked to the FX order flow.

Similarly to Bomfim (2003) we aim at modeling latent factors behind asset returns in terms of financial and economic fundamentals, rather than abstract distributional parameters, whether these are level, slope and curvature for the term structure models, or processes that define components of volatility matrices (as in many GARCH and stochastic volatility asset pricing models). We also share the view of Bomfim (2003) that the current and expected stance of monetary policy should be present among the explanatory factors behind asset prices. However, we extend Bomfim's paradigm by including other risk factors beside those directly linked to monetary policy, adding flexibility to the model.

The empirical modeling approach that we take was inspired by Ang and Piazzesi (2003) although we do not try to fit the observed yield curve. Instead, we estimate the pricing-kernel parameters that fit the returns of a number of basic infinite- or long-maturity assets. The reason is that we are looking for a possibly direct connection between macroeconomic risk factors, asset prices and the exchange rate. To view this connection through the prism of yield curve dynamics would be too circumspect for our purposes, since extraction of business-cycle information from the yield curve is likely to result in misspecification errors. In contrast, by working with a mapping from a space of unobserved factors to the space of basic assets, we are likely to capture the latent principal components responsible for the economic activity, inflation and monetary policy expectations in both modeled economies in a more direct way. We believe that a model constructed according to the above principles would contain less noise in the identified business-cycle position of the economy than most multi-factor yield-curve models in the literature.

Second, the econometric literature that implements ***empirical pricing--kernel models*** has provided a framework for linking the observed asset prices to the hidden risk factors in the equations for asset-return premia, without the need to use aggregate consumption data.[1] The basic idea of replacing the theoretical pricing kernel[2] equal to the real marginal rate of intertemporal consumption substitution, by its projection on the space of relevant risk factors, has been used in (Jackwerth, 2000), (Ait-Sahalia – Lo, 2002) or (Rosenberg – Engle, 2002), among others. The named authors call the result of the projection (the possibility to replace the original pricing kernel with the projected one is a simple consequence of the law of iterative expectations) the *empirical pricing kernel*. However, the main idea is of an abstract and general nature. So, it can be exploited in different situations irrespective of the chosen empirical context (e.g. in the present paper we apply it to an extended International CCAPM for a small open economy, which has not, to our knowledge, been done yet). Therefore, we prefer to use the term *projected pricing kernel*.

[1] Applying consumption-based CAPM directly to real financial data seems to be a dead end empirically – see (Rosenberg – Engle, 2002).

[2] The standard exposition of the consumption-based CAPM in pricing-kernel terms can be found in (Campbell et al., 1997) or (Cochrane, 2001).

Third, we exploit the ideas of the ***New Micro approach to exchange--rate economics***, by assigning the order flow in the FX market a prominent price-formation role. The best-known contributions to microstructural FX analysis (such as (Evans – Lyons, 2002)) provide a mechanism by which information dispersed among traders is being integrated by the FX order flow received by liquidity providers and impounded into the market price.[3] It turns out that the existing forex trading institutions function in such a way that all kinds of trader asymmetry are only observable through a limited number of sufficient statistics of market frictions. The latter grow with the transaction volume. Traditionally, FX market microstructure studies have relied on real-time (i.e. high-frequency) trade data to test their mo-dels. Nevertheless, many findings within this approach suggest that risks contained in the forex order flow have price-relevant consequences in macroeconomically relevant horizons as well (more on this can be found in (Lyons, 2001)). The idea was exploited in (Derviz, 2004b) to explain a part of the exchange-risk premium by learning from order flow and model-revision by FX dealers. In this paper, a similar mechanism, including integration of dispersed information, is implicit. We use its "reduced form" by introducing a deviation from the autarchic exchange rate by the actual price in the forex. The said deviation is due to the currency supply side risks and is captured by a sufficient statistic of the investor's dynamically adjusted outgoing FX order flow. This allows us to offer a natural explanation of a seemingly anomalous relationship between the spot rate, the forward rate and the order flow visible in the data. The corresponding equation is a part of a general no-arbitrage asset pricing equation system.

As is usually the case in the international asset-pricing literature, one needs to model both domestic and foreign-resident representative investor state prices to identify the equilibrium exchange rate parameters – see (Gourinchas – Tornell, 2002) for a recent example. Therefore, we start with an optimizing model for an investor resident in the "big country" (euro area) whose investment-opportunity set includes assets from the "small country" (Czech Republic) and derive the asset-pricing formulae in the big-country pricing-kernel terms. Then we reverse the perspective to obtain the same pricing formulae for the small country resident in his pricing-kernel terms. This is sufficient to obtain the autarchic exchange rate as the difference of the two pricing kernels.

We argue that modeling a representative big-country investor as a significant entity in the small-country asset markets is justified by the targeted empirical objective. It is a well-established fact of the Czech financial markets that non-resident securities traders constitute a prominent share of participants in most segments. (Informal evidence on other joining EU--member countries suggests very much the same.) Moreover, the role of non--resident traders in price formation is stable over time. This feature does

[3] Note that the popular notion of "fair" or "fundamental" price and its discovery in the course of trade, as often employed in microstructure finance of most securities, is poorly applicable to forex, since foreign cash is not an asset for which the contents of "superior private information" can be readily defined. Therefore, in the case of currency markets, the terminology has shifted toward terms like "dispersed information".

not contradict the possible saving-rate gap effect on investment flows the way international macroeconomics understands it, since it is mainly FDI that close the gap. The latter usually do not pass through public securities markets. There, the balance between resident and non-resident holdings did not change much during the period covered by our data.

Formally, the utilized assumption is that in almost all security markets there is one domestic and one foreign representative investor and there are no systematic unidirectional transfers of security holdings from one to another. The only exception is the FX market (which is at the center of our interest) where periods of big unidirectional cross-border order flows occur regularly.

The rest of the paper is organized as follows. Section 2 defines the theoretical model and derives the asset-pricing formulae to be implemented empirically. Section 3 formulates the statistical assumptions on the underlying risk factors and derives the model to be tested, in state-space form. Section 4 presents the estimation results. Section 5 sums up the findings, discusses the policy consequences and concludes the paper. The Appendix contains technical details of our estimation method.

2. The Economy

2.1 Definitions

We consider an investor (a financial institution as a typical example) acting in discrete time, for which an investment-opportunity set is constituted by domestic and foreign cash, as well as N_1 domestic and N_2 foreign assets. The risk-free domestic and foreign interest rates on cash holdings between dates t and $t + 1$ are denoted by r^0_{t+1} and r^i_{t+1}. A domestic asset d generates a stochastic sequence $(\Gamma^d_t)_{t\geq 0}$ of cash incomes (dividends or coupons if fixed income) in the domestic currency ($d = 1,..., N_1$), and a foreign asset f generates a similar sequence $(\Gamma^f_t)_{t\geq 0}$ in the foreign currency ($f = 1,..., N_2$). One share of domestic asset d costs P^d_t units of domestic currency in period t, and, analogously, P_t^f is the period t-price in foreign currency of one unit of foreign asset f. The end of date t holdings of asset d (f) are denoted by x^d_t (x_t^f), and the cash holdings – by x_t^0 and x_t^i. Finally, the period t-domestic currency price of one foreign currency unit (the nominal exchange rate) will be denoted by S_t. The investor faces the uncertainty associated with asset earnings, Γ^d and Γ^f, and prices, P, P^d, P^f, S ($d = 1,..., N_1$, $f = 1,..., N_2$, P is the domestic-consumption price level to be introduced shortly).

The nominal purchase/sale order volume of security d (f) in period t will be denoted by φ^d_t (φ^f_t), and symbol φ_t^i will stand for the foreign cash purchase/sale order volume in period t.

2.1.1 Transaction Costs

The asset market transaction-related frictions to be defined below are the key element of the model extending its validity beyond the standard consumption-based CAPM.

The investor entering the market for a given security as a market user faces a set of liquidity-providers (market makers in the quote-driven trading mechanism and brokers in the order-driven mechanism). In aggregate, these liquidity providers present the investor with a *pricing schedule*, which is an increasing function of his order. That is, a selling investor gets less than the mid-price, the price reduction increasing in the sale volume, and a purchasing investor pays more than the mid-price, the price increase growing with the purchase volume.

Formally, if, at the start of period t, the investor decides to purchase φ^d_t, he pays $P^d_t j^d(\varphi^d_t)$, where P^d_t is the unit price and $j^d(\varphi^d_t)$ is a transaction factor. The latter is generated by a strictly increasing and convex function j^d, with $j^d(0) = 0$, $j^{d'}(0) = 1$. That is, $P^d_t[j^d(\varphi^d_t) - \varphi^d_t]$ is a "fee", or premium paid (in domestic cash) to an intermediary. This premium is increasing with the order volume φ^d_t. If the amount $-\varphi^d_t > 0$ is sold, the investor receives $-P^d_t j^d(\varphi^d_t)$, which is less than $-P^d_t \varphi^d_t$, with the intermediation premium being, again, equal to $P^d_t[j^d(\varphi^d_t) - \varphi^d_t]$. That is, transaction costs are positive for all non-zero trades. Analogously, for transactions in a foreign asset f, a similar strictly increasing and convex transaction factor j^f is defined, and a fee $P^f_t[j^f(\varphi^f{}_t) - \varphi^f{}_t]$ must be paid in the foreign currency. The same is true for the transactions between the domestic and foreign cash (the FX market), where the corresponding factor is denoted by j. For a net purchase of $\varphi^i{}_t$ foreign currency units at date t, the investor pays the fee $S_t[j(\varphi^i{}_t) - \varphi^i{}_t]$.[4]

The above construction corresponds to aggregating the liquidity providers for a given security (market makers and brokers) in an imaginary global limit order book. Then, the resulting order execution costs for a market user (i.e. the investor we are modeling) can be approximated by a convex transaction function of the order size, as defined above. This construction of the market-wide pricing schedule has been used in market microstructure models at least since Kyle (1985). The property is also sufficiently theoretically supported by microstructure finance literature. For example, a specialist's pricing schedule as an increasing function of order size has been derived from first principles for both a monopolist and a competitive market maker in (Glosten, 1989), among others.

2.1.2 Preferences

The investor, of whom we think as residing in the home country, derives utility from the real consumption or dividend, rate c withdrawn from the domestic real cash balances. At every time moment, this utility is in-

[4] To see the intuition underlying the above definition, let us first consider a quote-driven trading mechanism. For most securities (including FX), such markets have a convention of quoting bid and ask for *standard amounts*. Everything beyond the standard is either charged a different price with a premium for the market maker as a routine or has to be negotiated separately, if accepted at all. In any case, a market maker charges a higher price for a higher purchase order (pays a lower price for a higher sale order) as protection against open position risk. Although economies of scale may exist in terms of the customer-base size of a given dealer (i.e. dealers benefit from handling a large number of orders), there is no such thing in terms of the *order size* itself. In the order-driven market (broker, limit order book), an increase in market order size can have one of two consequences. The trader either incurs an execution delay and additional costs (the order has to be split with later segments facing a higher price in a changed limit order book) or needs to "walk the book" (i.e. execute outstanding parts of the order at the second-best, third-best, etc. limit price available). Both variants mean execution expenditure growing with volume.

fluenced by the investor's solvency, expressed in real terms by the liquid balance $l = (x^0 + Sx^i)/P$, where P is the domestic price level. The period utility u is a function of two arguments, l and c, smooth, strictly increasing and strictly concave in each of them. To guarantee an internal solution to the investor optimization problem, at the same time excluding cash debt accumulation, we assume, for all c and l, that:

$$u_c(l,+0) = \frac{\partial u}{\partial c}(l,+0) = +\infty, \lim_{l \to -\infty} u(l,c) = -\infty, u_l(+0,c) = \frac{\partial u}{\partial l}(+0,c) = +\infty$$

2.2 Investor's Optimization Problem

The investor has an infinite horizon and maximizes the expected utility ($\delta \in (0,1)$ is the time preference factor)

$$\sum_{\tau \geq 0} \delta^\tau E\left[u\left(\frac{x_\tau^0 + S_\tau x_\tau^i}{P_\tau}, c_\tau\right)\right] \tag{1}$$

subject to the constraints ($t \geq 0$)

$$x_{t+1}^0 = (1+ r_{t+1}^0)\, x_t^0 + \sum_{d=1}^{N_1} x_t^d \Gamma_t^d - P_{t+1}c_{t+1} - S_{t+1} j(\varphi^i{}_{t+1}) - \sum_{d=1}^{N_1} P_{t+1}^d j^d(\varphi_{t+1}^d) \tag{2.1}$$

$$x_{t+1}^i = (1+ r_{t+1}^i) x_t^i + \sum_{f=1}^{N_2} x_t^f \Gamma_t^f + \varphi_{t+1}^i - \sum_{f=1}^{N_2} P_{t+1}^f j^f(\varphi_{t+1}^f) \tag{2.2}$$

$$x_{t+1}^d = x_t^d + \varphi_{t+1}^d, \qquad d = 1, ..., N_1 \tag{2.3}$$

$$x_{t+1}^f = x_t^f + \varphi_{t+1}^f, \qquad f = 1, ..., N_2 \tag{2.4}$$

The initial cash and asset holdings x_0^0, x_0^i, x_0^d, x_0^f are given. The decision variables of the investor are the trajectories $t \mapsto c_t$, $t \geq 0$, $t \mapsto \varphi^i{}_t$, $t \mapsto \varphi_t^d$, $t \mapsto \varphi_t^f$, $t \geq 1$, for all d and f. More specifically, at the end of period t and the beginning of period $t + 1$, the investor knows the realizations of P_{t+1}, r_{t+1}^0, r_{t+1}^i, S_{t+1}, P_{t+1}^d, P_{t+1}^f, Γ_t^d, Γ_t^f (plus the histories of these variables and the distributions of their uncertain values one period ahead) and has to decide about the values of c_t, φ_{t+1}^d, φ_{t+1}^f, φ_{t+1}^i, $d = 1,...., N_1$, $f = 1,...., N_2$.

Alternatively, one can express c_τ by using (2.1), φ_τ^i by using (2.2) and φ_τ^d, φ_τ^f from (2.3), (2.4), and substitute them into (1). Then the problem becomes that of unconstrained optimization with respect to the asset holding vector path $t \mapsto x_t = [x_t^0, x_t^i, (x_t^d)_{d=1}^{N_1}, (x_t^f)_{f=1}^{N_2}]^T$ $t \geq 1$.

Observe that due to the liquidity-dependence of the period utility in (1) the transversality conditions on the components of x are not needed in this optimization problem: the Ponzi-like behavior is prohibited by the conditions on the utility-function behavior under big negative cash values. Similarly, unlimited short-selling of any asset is prohibited by convex transaction factors j.

Also note that equations (2) allocate the home asset transaction fees to the domestic cash account of the investor and the foreign-asset transaction fees to the foreign cash one. For the FX trade, the fee is subtracted from the domestic cash account if the investor resides in the home country (we would have modeled its subtraction from the foreign cash account in the case of a foreign resident). Specifically, (2.1) contains a reduction in domestic cash holdings corresponding to φ^i_{t+1}, namely the term $-S_{t+1}j(\varphi^i_{t+1})$. The latter can be decomposed as $-S_{t+1}\varphi^i_{t+1} - S_{t+1}[j(\varphi^i_{t+1}) - \varphi^i_{t+1}]$, the second term in this expression being the transaction fee. Thus, the domestic investor, when purchasing φ^i_{t+1} units of foreign currency, pays the amount in domestic cash determined by the current nominal exchange rate, plus a fee. (Conversely, when foreign cash is sold, the fee is subtracted from the sale revenue.) A similar interpretation can be given to other asset trade expenditures.

To formulate the first order conditions of optimality for the investor's policies, we introduce the *marginal transaction functions* $\varphi \mapsto h^d(\varphi) = \dfrac{dj^d}{d\varphi}(\varphi)$, $\varphi \mapsto h^f(\varphi) = \dfrac{dj^f}{d\varphi}(\varphi)$, $\varphi \mapsto h(\varphi) = j'(\varphi)$. They are increasing (since js are convex) and equal to unity at the origin. They give rise to the investor's asset demand schedules (see later).

Further, we define the *autarky asset prices* (the reason for the name is that these are the prices that prevail in markets where investors choose to place zero-size orders) as follows:

$$X_t^d = P_t^d h^d(\varphi_t^d),\ \ X_t^f = P_t^f h^f(\varphi_t^f),\ \ \ X_t = S_t h(\varphi_t^i)$$

Their role in the model will become clear shortly (see Proposition 1 and the discussion thereafter).

Finally, we introduce the pricing kernel in the usual way:

$$M_t^{\tau+1} = \frac{\delta P_t u_c(l_{t+1}, c_{t+1})}{P_{t+1}u_c(l_t, c_t)},\ m_{t+1} = \log M_t^{\tau+1}$$

and also, for notational convenience, we define an auxiliary symbol for the marginal rate of substitution between consumption and liquidity:

$$L_t = \frac{u_l(l_t, c_t)}{u_c(l_t, c_t)}$$

The investor's first order conditions of optimality are given by:

Proposition 1 *The first order conditions of optimality for the investor's optimization problem* (1), (2), *are given by a sequence of four equations* (t = 0, 1, ...).

$$E_t\left[M_t^{t+1}\frac{\Gamma_t^d + X_{t+1}^d}{X_t^d}\right] = 1,\ d = 1, ..., N_1,\ E_t\left[M_t^{t+1}\frac{X_{t+1}}{X_t}\frac{\Gamma_t^f + X^f_{t+1}}{X^f_t}\right] = 1$$

$$f = 1, ..., N_2 \tag{3a}$$

$$E_t\left[M_t^{t+1}\right] = \frac{1 - L_t}{1 + r_{t+1}^0},\ E_t\left[M_t^{t+1}\frac{X_{t+1}}{X_t}\right] = \frac{1 - L_t / h(\varphi_t{}^i)}{1 + r_{t+1}^i} \tag{3b}$$

As usual, E_t denotes expectation conditional on the information available up to period t.

The proof is based on standard dynamic optimization considerations and is, therefore, omitted for brevity – cf. (Derviz, 2004a), where the argument for a similar problem is provided.

Formulae (3) can be regarded as no-arbitrage pricing conditions. However, differently from the traditional asset pricing model, they are formulated in terms of autarchic prices X instead of the actual prices P and S. Therefore, the price system X can be viewed as the underlying/implicit price process in a Walrasian market, for which the standard pricing-kernel relationships are valid exactly. Since the actual prices become equal to the implicit ones if and only if there are no transactions in the markets ($\varphi^I = \varphi^d = \varphi^f = 0$), X can be called autarchic prices, as mentioned earlier.

2.2.1 Actual Price, Autarky Price and Order Flow

An autarchic asset price can be also regarded as the intercept of the investor's current demand schedule for this asset. We'll illustrate this with the FX market example which is in the center of attention in the present paper.

By rewriting the definition of X as $S_t = \frac{X_t}{h(\varphi_t{}^i)}$, we can regard the latter equation as the definition of the inverse demand curve in period t. When the current price (exchange rate) is below the autarky value, the investor optimally purchases foreign currency, and *vice versa*. The price is then reverted back to its autarky value, which is the one that obeys the standard no-arbitrage rules of a hypothetical perfectly frictionless market.

What is the reason for the actual price to deviate from the autarchic one? The general answer is that it must always be the case when the representative investor assumption with instantaneously clearing markets is not justified. We believe that the forex is an archetype of such a market. In this paper, we model a collection of financial market segments that include, beside the spot market for the CZK/EUR pair, a number of assets, both Czech and European, whose prices we believe may be related to the Czech koruna exchange rate. We are not modeling the totality of the euro-area securities markets, but only its selected elements with a bearing on the financial and real developments in the Czech economy. (For formal reasons, we have also included one asset on the euro-area side that should proxy all Czech-unrelated uncertainties.) So, our model is of a partial equilibrium nature. In this section, the case of the big-country resident investor is selected for definiteness. At the estimation phase (Section 4), this perspective will be alternated with the reverse one (a small-country resident investor) in order to identify more of the model parameters, the autarchic FX return $\log \frac{X_{t+1}}{X_t}$ in particular. The investor is allowed to have external counterparties in the koruna/euro market, and we also assume that this investor does not himself make the market in the Czech koruna. Therefore, it is natural to assume the existence of a non-trivial active order flow from this investor outside (directed towards koruna FX dealers and/or Czech monetary authorities). This flow is represented by variable φ^i in the model. As regards the other asset markets under consideration, it will be assumed that the modeled investor is sufficiently representative in the corresponding market segments for the non-Walrasian phenomena to be neglected.

It will be convenient to rephrase the first order conditions (3) in terms of continuously compounded returns. Specifically, let the *domestic currency autarchic* one-period returns on, respectively domestic asset d and foreign asset f, between dates t and $t + 1$ be defined as:

$$y^d_{t+1} = \log \frac{\Gamma^d_t + X^d_{t+1}}{X^d_t}, \quad y^f_{t+1} = \log \frac{X_{t+1}(\Gamma^f_t + X^f_{t+1})}{X_t X^f_t}$$

Similarly, $\rho^0_{t+1} = \log(1 + r^0_{t+1})$, $\rho^i_{t+1} = \log(1 + r^i_{t+1})$ will be the continuously compounded one-period risk-free interest rates at home and abroad. Finally, let z_{t+1} be the one period autarchic return on foreign cash: $z_{t+1} = \log \frac{X_{t+1}}{X_t}$.

Conditions (3a) can be now stated uniformly as:

$$E_t\left[e^{m_{t+1} + y^j_{t+1}}\right] = 1, j = 1, ..., N = N_1 + N_2 \tag{4}$$

So, with the exception of the risk-free rates, equations (3b) only contain unobservable variables. Modeling the cash-liquidity marginal substitution rate would mean opening an additional dimension of the analysis with no direct benefit for the objective of this study. Instead, we choose to use only a consequence of (3b) in terms of the pricing kernel and the FX order flow that can be obtained by substituting L_t away. The substitution leads to the following equation:

$$h(\varphi_t{}^i)\left\{1 - E_t\left\lfloor e^{\rho^i_{t+1} + m_{t+1} + z_{t+1}}\right\rfloor\right\} = 1 - E_t\left\lfloor e^{\rho^0_{t+1} + m_{t+1}}\right\rfloor \tag{5}$$

Equations (4) and (5) form the foundation of our empirical model. We shall assume that there are n independent Gaussian risk factors jointly driving the returns on the selected securities and the FX order flow. The dynamics of these unobserved risk factors will be estimated together with the signal equations to be obtained from (4), (5). The difference of this equation system from the textbook (international) consumption CAPM is the market incompleteness due to a separate source of forex risk.

3. Implementation in State Space Form

The asset pricing formulae following from the investor's first order conditions of optimality (3) (Proposition 1) will now serve as a starting point for representing the model in the state space form. The individual equations in (3) will be specialized to observation equations after we have made specific assumptions about the statistical properties of uncertainty driving the model variables. At the same time, pinning down the structure of uncertainty will be paramount to formulating the state variable dynamics of the model – cf. (7) and (10) below.

The observation variables of our model are the traded asset yields (cf. $y^1,\ldots,y^N$, as defined in Subsection 2.3, which are autarchic yields of the same assets), and the one-period interest rate (as appearing in (5)) differential $f = \rho^0 - \rho^i$ (see also (14) below). The state process will be assumed multivariate autoregressive of order one, with Gaussian innovations. Restrictions on observation equation coefficients will follow from the no-arbitrage asset pricing conditions (3) (see Proposition 2). Formal specification and derivation follows.

We start by stating the equations for the autarchic yields formally as:

$$y^j_{t+1} = a^j_0 + a^j_1 x_t + A^j x_{t+1}, j = 1, \ldots, N \tag{6}$$

Here, $a_0 = [a^1_0, \ldots, a^N_0]^T$ is an $N \times 1$-vector of intercepts, a_1 and A are $N \times n$-matrices of coefficients with rows $a^j_1 = \lfloor a^j_{11}, \ldots, a^j_{1n} \rfloor$ and $A^j = \lfloor A^j_1, \ldots, A^j_n \rfloor$ respectively. The n-dimensional vector x of unobserved state residuals follows the VAR(1)-process:

$$x_{t+1} = bx_t + B\varepsilon_{t+1} \tag{7}$$

Coefficient matrices b and B in (7) are of size $n \times n$. Process ε is an n-dimensional vector of mutually independent standard normal errors. In general, $n \neq N$, and, if there is a reason to assume, e.g. cyclical components in the observations, one will need to take $n > N$.

The number of modeled parameters in (6) and (7) can be reduced. Namely, without loss of generality, we may consider a non-singular matrix B (otherwise, there would be too many states) and put $x_t = Bu_t$ for all t (this is equivalent to assuming $B = \mathrm{I}$ – identity matrix). Then:

$$u_{t+1} = \Phi u_t + \varepsilon_{t+1},\ \Phi = B^{-1}bB \tag{8}$$

An additional unobserved variable to be used in the sequel is the log of the one-period pricing kernel $m_{t+1} = \log M^{t+1}_t$. In terms of state process u, it will be expressed as:

$$m_{t+1} = c_0 + c_1 u_t + Cu_{t+1} \tag{9}$$

Here, c_0 is a scalar constant, whereas c_1 and C are row vectors of dimension n.

The equation system (6), (8), (9) is not the definitive representation to be estimated. First of all, one must incorporate the coefficient restrictions following from the no-arbitrage pricing conditions (4), (5) (a proof can be obtained by standard matrix algebra – cf. (Derviz, 2004a)).

Proposition 2 *The no-arbitrage pricing conditions* (4) *for the asset return model defined by* (6), (8) (9) *above are equivalent to the following constraints on the model coefficients*:

$$a^j_0 = -c_0 - \frac{|C + A^jB|^2}{2},\ a^j_1 = -c_1 - (C + A^jB)\Phi, j = 1, ..., N \tag{10}$$

Note that, whereas the first equality in (10) is scalar, the second one is for N-dimensional row vectors.

Put $\gamma^j = C + A^jB$. It is easily checked that the no-arbitrage pricing conditions (10) of Proposition 2 imply the following equations for the autarchic yields:

$$y^j_{t+1} = -c_0 - \frac{|\gamma^j|^2}{2} - (c_1 + \gamma^j\Phi)\, u_t + (\gamma^j - C)\, u_{t+1}$$

$$= - c_0 - \frac{|C + q^j|^2}{2} - \mu u_t + q^j \varepsilon_{t+1} = -m_{t+1} - \frac{|\gamma^j|^2}{2} + \gamma^j \varepsilon_{t+1}, j = 1, ..., N \tag{11}$$

where $\mu = c_1 + C\Phi$ and $q^j = \gamma - C$.

With the chosen normal state variable errors, implementation of conditions (4) has lead to relatively simple return formulae (11) with linear-quadratic coefficient constraints. Handling non-linearity in (5) is less straightforward. We will need to make a natural assumption that the autarchic FX return z has the same structure of risk-factor dependence as other autarky returns:

$$z_{t+1} = \varsigma_1 u_t + \varsigma u_{t+1} = \xi u_t + \varsigma\varepsilon_{t+1} \tag{12}$$

with $\xi = \xi_1 + \xi\Phi$. Observe the missing constant term in (12): we do not assume any equilibrium trends in the autarchic exchange rate.

Here we introduce the certainty-equivalence shorthand notation:

$$\eta_t = c_0 + (c_1 + C\Phi)u_t + \frac{|C|^2}{2},\ \eta^*_t = c_0 + (c_1 + \varsigma_1 + (C + \varsigma)\Phi)u_t + \frac{|C + \varsigma|^2}{2} \tag{13}$$

Then (5) can be rewritten as:

$$h(\varphi_t{}^i) = \frac{e^{\rho^0_{t+1}+\eta_t} - 1}{e^{\rho^i_{t+1}+\eta^*_t} - 1} \tag{14}$$

Since one needs a linear structure in the unobserved states to apply the conventional Kalman filter technique, (14) must be further transformed.

Qualitatively, this equation connects *j*the aggregate FX order flow φ^i with the risk-adjusted interest rate difference $\rho^0 - \rho^I + \eta - \eta^*$, plus higher order terms. Therefore, the easiest linearized version of (14) to use would be:

$$f_t = \frac{|C + \varsigma|^2}{2} - \frac{|C|^2}{2} + (\varsigma_1 + \varsigma\Phi)u_t + \alpha\varphi_t{}^i = \varsigma\left(\frac{\varsigma}{2} + C\right) + \alpha\varphi_t{}^i + \xi u_t \tag{15}$$

where $f = \rho^0 - \rho^i$ is the forward exchange premium/discount and α a positive constant. This would correspond to the marginal transaction function in the forex to be of the time-dependent form

$$h(\varphi_t{}^i) = H(t,\varphi_t{}^i) = \frac{1 - e^{-\rho^0_{t+1} - \eta_t}}{e^{-\alpha\varphi^i_t} - e^{-\rho^i_{t+1} - \eta^*_t}} \tag{16}$$

One can easily check that the marginal transaction function defined in this way has the right properties, i.e. is increasing, strictly positive for admissible realizations of the model variables, and takes the value of unity at the origin.

Equation (14) relates the aggregate FX order flow (the modeled investor--initiated net FX purchases if positive, sales if negative) and the forward premium in a seemingly counterintuitive way. Namely, high forward premia (i.e. a relatively high domestic short interest rate, risk-adjustment taken into account) correspond to a flow out of the domestic currency, and *vice versa*. On the other hand, what may appear unnatural from a naive portfolio-adjustment point of view is in line with the observed reality. The data (*Figure 1*) show the existence of prolonged periods of positive correlation between the strengthening foreign (in this case meaning Czech) currency, the growing home-foreign interest rate differential and the order flow into the foreign cash.

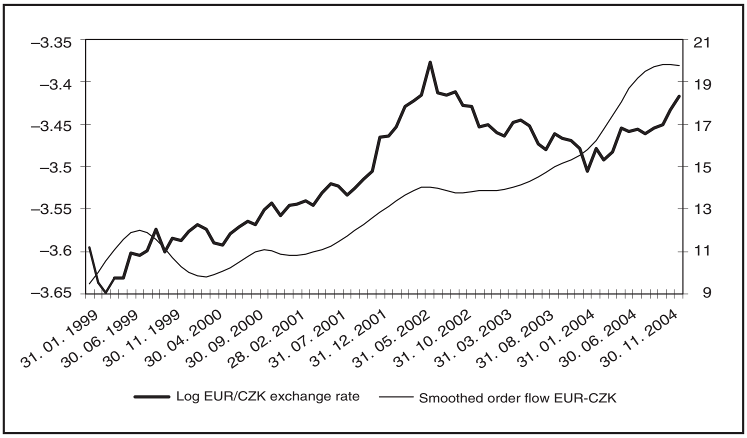
In the model, the observed sign of the OF-dependence is due to the fact that high forward premia correspond to high shadow/autarchic foreign currency values. (Recall that a risk-adjusted UIP holds for the shadow prices in the present model.) Shadow currency values above the actual ones induce an order flow into it, reverting the actual exchange rate back to its autarky level. Simultaneously with the accomplishment of this reversion, the forward premium falls to its neutral level and the positive order flow dies out. Thus, the model offers a qualitative explanation of the positive relation between the order flow and the forward premium observed in the data. Quantitatively, the FX risk premium coming out of (14) is non-stationary. Beside the time dependence contained in the order flow, the residuals of the form ξu_t are multivariate-autoregressive, due to (9), and conditionally correlated with the residuals in the asset-return equations, due to (11).

3.1 Identification of the Autarky Exchange Rate

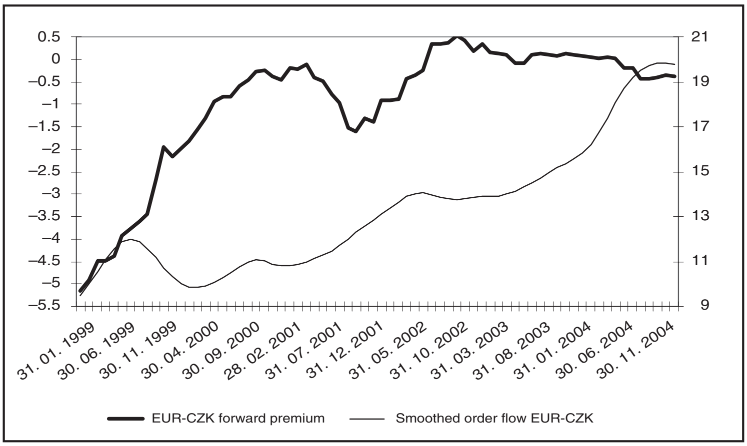
If one limits attention to the vantage point of the domestically based investor, then the autarchic FX return, z, described by (12), remains unidentified. One way to make identification possible is to take into consideration

FIGURE 1 Smoothed Proxies of the Euro-Czech Koruna Order Flow and the CZK/EUR Market

a) OF Proxy and the Spot EUR/CZK Rate



b) OF Proxy and the 1M EUR/CZK Forward Premium



the perspective of a foreign investor with the same information and investment-opportunity set as the domestic one defined earlier. In the FX order flow model, the foreign investor may be assumed to face a similar supply uncertainty. The model is, actually, symmetric with respect to the investor residency, so that the first order optimality conditions of Subsection 2.3 can be obtained for the foreigner by a simple adjustment of notations. In addition, our focus on macroeconomic implications of the forex risks suggests we may consider the non-Walrasian effects in other asset markets as purely random: $y^j = r^j - \nu^j$, where r^j is the actual return on asset j and ν^j is a random observation error. Then, the autarky exchange rate identification can be achieved by making the following:

Assumption 1 *The foreign investors are identical and face the same underlying risks, characterized by* (7), *as the representative domestic investor. There is no trend transfer of basic asset holdings from the foreigners to either the domestic investor or any third party. That is, the difference between the autarchic and observed basic asset prices is purely random.*

It can be shown that one case in which Assumption 1 is legitimate is when the foreign investor weight in the basic asset markets is negligible compared to the domestic resident. If their sizes are comparable, Assumption 1 is still justified provided that the FX supply uncertainty is identical for investors of both residencies and is generated by third parties (e.g. the central banks or inventory traders that we do not model explicitly).

Now, (4) can be restated as the no-arbitrage pricing equations for the actual instead of autarchic returns. They would contain, beside the error terms ν^j, just one unobservable variable (the pricing kernel logarithm m).

As follows from the mentioned symmetry of the model, the foreign investor faces the basic asset one-period autarchic returns:

$$y^{d*}_{t+1} = \log \frac{X_t \left(\Gamma^d_t + X^d_{t+1}\right)}{X_{t+1} P^d_t}, \; y^{f*}_{t+1} = \log \frac{\Gamma^f_t + X^f_{t+1}}{X^f_t} \tag{17}$$

Accordingly, the autarchic FX return z satisfies the equalities $y^{d*} = y^d + z$, $y^f = y^{f*} + z$ for all d and f. Given the no-arbitrage pricing conditions (11), the assumed underlying risk structure and Assumption 1, the above equalities all follow from the single condition on the pricing kernels for the two investors:

$$z_t = m_t - m^*_t \tag{18}$$

or, in terms of the coefficients:

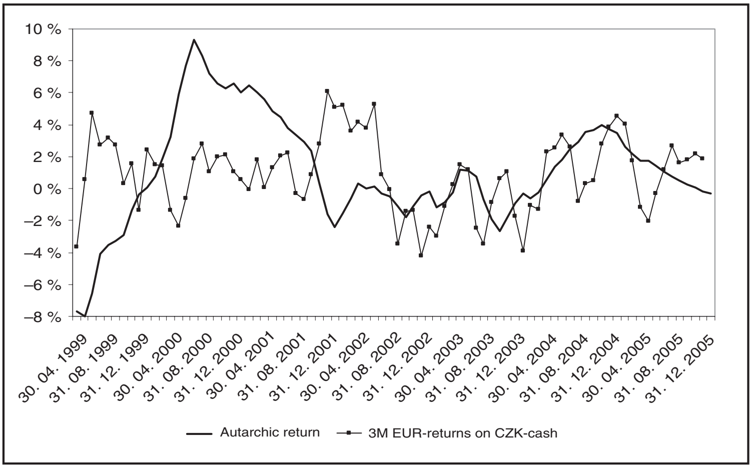
$$\zeta = C^* - C, \qquad \varsigma_1 = c^*_1 - c_1 \tag{18*}$$

Here, asterisks denote the pricing-kernel coefficients of the foreigner. Therefore, the autarchic exchange-rate coefficients are available if the pricing kernel coefficient estimates of both the domestic and the foreign investor can be obtained. *Figure 2* features the actual and the autarchic monthly EUR/CZK returns, estimated by means of the procedure discussed below.

4. Estimation

The estimation procedure for the model defined in the previous two sections has to be adapted to the short length of available time series covering the Czech capital market, as well as the substantial structural changes that this market underwent at the beginning of 1999. Particularly, the features of the Czech koruna FX spot segment changed dramatically after the withdrawal of most short interest rate speculators, discouraged by

FIGURE 2 Actual and Autarchic Monthly EUR/CZK Returns (3M-smoothed)



the decrease of the koruna-euro interest rate differential. So, given that this study has been conceived so as to be usable on a recurrent basis with the possibility to update the sample continuously, it would make little sense to analyze the data from earlier periods.

As a proxy for the cross-border euro-koruna order flow, we use the data currently collected in monthly periodicity on Czech koruna demand deposit accounts of non-residents (reported by banks to the balance of payments statistics). We discuss the reason in the next subsection.

4.1 Data

Our sample is of monthly data between 1:1999 and 9:2005. An extended candidate list for representing the observation variables for macro risks has been considered. Among other things, we looked at measures of real activity (such as real industrial production index and real construction index), inflation, market confidence indicators and interest rates. Based on the factor analysis, we decided to use the following 10 macroeconomic time series (five for the Czech Republic and five for Germany) as observables w_t related to the underlying normalized macroeconomic risk factors u_t. The chosen series are: the industrial production index, the harmonized index of consumer prices (HICP), the total economic sentiment index, 1M money market interest rate and the yield spread between the 10Y government bonds yield and the 1M money market rate. The production and price index variables have been transformed to percent changes to eliminate non-stationarity.

The Czech asset returns are represented by two time series: returns on the global index PX50 and the financial sector index BI15.[5] German asset

[5] Since sector index BI15 ceased to be published by September 2005, its returns have to be replaced by the average of returns of Komercni banka and Erste Bank stocks for subsequent exercises with actualized time series.

returns are represented by time series of returns of DAX and DAX finance. Depending on the reference currency of the modeled investor, ex post returns on the other country assets are exchange-rate adjusted.

The sample size is 84 months, the size of the observation vector is 19 (18 in the benchmark model without the FX order flow factor) and the number of parameters is 314 (309 in the benchmark). Thus the number of the degrees of freedom of the presently estimated state space model is $1282 = 84 \cdot 19 - 314$ (or $1203 = 84 \cdot 18 - 309$ for the benchmark). So, even with the described sample the Kalman filter estimation results admit a meaningful interpretation. On the other hand, given the sample length, inclusion of a bigger number of basic assets would not add much value to the analysis.

An aggregate order flow of the type appropriate for the present model is usually unobservable. Therefore, we are using a proxy in the form of non-resident Czech koruna-denominated demand deposits in the Czech banking system, as well as resident euro-denominated demand deposits in the same banks, for each reported month. Formally, the received FX cross-border order flow is mostly accounted for in the bank statistics as koruna transactions with non-residents, whereas active (outgoing) FX order flow of the same banks falls into the category of foreign-currency transactions. This is why the banking statistics are able to produce a time series with a positive correlation with the net FX cross-border order flow into and out of koruna.

Formally, the choice of a proxy for the order flow means that linearization of the no-arbitrage condition (3b) is performed with the marginal transaction function H defined in (16), with the chosen proxy replacing φ^i. Since both the actual order flow and the marginal transaction function are unobservable, this formal substitution does not impose additional restrictions on the model.

4.2 Empirical Model

For the ease of reference, we collect here the elements of the model to be estimated (cf. (8), (11) and (15) of Section 3). Below, (19) and (20) are the state equations and (21)–(24) are the measurement equations.

$$u_{t+1} = \Phi u_t + \varepsilon_{t+1} \tag{19}$$

$$m_t = c_0 + c_1 u_{t-1} + C u_t \qquad m_t^* = c_0^* + c_1^* u_{t-1} + C^* u_t \tag{20}$$

$$y_{t+1}^{d_j} = -m_{t+1} - \frac{\|\gamma^j\|^2}{2} + \gamma^j \varepsilon_{t+1} + \nu_{t+1}^j \qquad y_{t+1}^{d_f} = -m_{t+1}^* - \frac{\|\gamma^j\|^2}{2} + \gamma^j \varepsilon_{t+1} + \nu_{t+1}^{j*} \tag{21}$$

$$z_{t+1} = \xi u_t + \zeta \varepsilon_{t+1} \tag{22}$$

$$\rho_{t+1}^0 - \rho_{t+1}^i = \zeta \left(\frac{\zeta}{2} + C \right)^T + \alpha \varphi_{t+1} + \xi u_{t+1} \tag{23}$$

where u_t is a vector of macroeconomic factors, that follows an autoregressive relation (19), with i.i.d. innovations ε_t, that have a unit covariance ma-

TABLE 1 State Variable Decomposition of the Macro Risks

Normalized risk factors →	$u_1(-1)$	$u_2(-1)$	$u_3(-1)$	$u_4(-1)$	$u_5(-1)$	$u_6(-1)$	$u_7(-1)$	$u_8(-1)$	$u_9(-1)$	$u_{10}(-1)$	**Contribution to the conditional autarchic FX return variance, %**
Macro risk factors ↓											
IPI GE	0.9121	0.0074	-0.1066	-1.27	1.2722	0	0	0	0	0	1.5
Sent GE	0.0074	0.5418	0.0505	-1.0978	1.4068	0	0	0	0	0	12.6
HICP GE	0.0031	-0.0031	0.7631	0.2302	0.0703	0	0	0	0	0	3.1
1M EURIBOR	0.0016	-0.0009	0.0808	0.8499	0.0595	0	0	0	0	0	7.3
EUR yield spread	-0.0058	-0.0034	0.0399	0.094	0.7188	0	0	0	0	0	16.4
IPI CZ	6.9356	0.0798	0	0	0	0.9446	-0.0085	-0.0418	0.0468	-0.0141	0.3
Sent CZ	4.6057	-0.0919	0	0	0	0.0119	0.0018	0.0059	-0.0026	0.0002	23.0
HICP CZ	0	0	0.3977	0	0	0	0	0.5843	0.19	0.0092	8.6
1M PRIBOR	-0.0994	0.0064	0	0.3319	0.214	0.1358	-0.0017	0.1556	0.6168	-0.1169	23.1
CZK yield spread	-0.9276	0.0166	0	0.0847	0.323	0.2598	0.0072	-0.2233	0.0163	0.4856	3.1

The normalized risk variables are multivariate autoregressive: $u_j = \sum_{i=1}^{7} \Phi^i_j u_i(-1) + \varepsilon_j$, with ε being standard normal independent disturbances.

The variable names stand for: IPI GE – German industrial production index growth; Sent GE – economic sentiment in Germany; HICP GE – German harmonized index of consumer price inflation; EUR yield spread – difference between 10Y German government bond yield and 1M EURIBOR; IPI CZ – Czech industrial production index growth; Sent CZ – economic sentiment in the Czech Republic; HICP CZ – Czech harmonized consumer price inflation; CZK yield spread – difference between 10Y Czech government bond yield and 1M PRIBOR.

The last column shows relative contributions in percent of the individual macro factors to the variance of the FX return conditional on past information, based on estimated coefficient values.

trix. We do not observe these normalized risk factors, but rather the vector of linear combinations $w_t = Bu_t$ which consists of the measured macroeconomic variables listed in 4.1 above. The parameters appearing in (19)–(23) obey the restrictions $\xi = \zeta_1 + \zeta\Phi$, $\zeta = C^* - C$, $\zeta_1 = c_1^* - c_1$. Recall that m, m^* are the logs of the pricing kernel of, respectively, the EU-based and the Czech-based investor. The missing constant term in (22) implies that $c_0 = c_0^*$.

Recall that asset returns from the domestic perspective are summarized in the vector $y_t^d = [y_t^1 \cdots y_t^J]^T$ (similarly y_t^f is the vector of asset returns from the foreign perspective). Equation (21) reflects the fact that the observed yields are measured with noise: ν_{t+1}^j is the EU investor's measurement error of the yield of asset j, while ν_{t+1}^{*j} is the Czech investor's measurement error of the return of the same asset; ν_{t+1}^j and ν_{t+1}^{*j} may be correlated.

In equation (23), the interest rate differential is denoted by $\rho_{t+1}^0 - \rho_{t+1}^j$ and the FX order flow – by φ_t. For the latter, we observe noisy proxies φ_t^i, φ_t^{i*} only, these are linked to the true order flow as follows:

$$\varphi_t^i = \widetilde{\alpha}_0 + \widetilde{\alpha}\,\varphi_t + \nu_t^{\varphi} \qquad \varphi_t^{i*} = \widetilde{\alpha}_0^* + \widetilde{\alpha}^*\varphi_t + \nu_t^{*\varphi} \tag{24}$$

where ν_{t+1}^{φ} and $\nu_{t+1}^{*\varphi}$ are the measurement noises of the order flows from

TABLE 2 Pricing Kernel and Autarchic FX Return Coefficients

Parameter	EU-investor		CZ-investor		Autarchic exchange rate		
	Estimated value	*P*-value	Estimated value	*P*-value	Parameter	Estimated value	*P*-value
$\Lambda(1)$	–0.041	0.000	–0.034	0.000	$\xi(1)$	–0.134	0.09
$\Lambda(2)$	0.000	0.841	0.232	0.000	$\xi(2)$	–0.846	0.04
$\Lambda(3)$	0.000	0.712	0.000	0.791	$\xi(3)$	–0.029	0.22
$\Lambda(4)$	–0.016	0.911	0.003	1.000	$\xi(4)$	0.043	0.02
$\Lambda(5)$	–0.007	0.871	0.031	1.000	$\xi(5)$	0.019	0.23
$\Lambda(6)$	–0.045	0.036	–0.013	0.271	$\xi(6)$	0.055	0.15
$\Lambda(7)$	–0.003	1.000	–0.008	0.000	$\xi(7)$	–0.003	0.09
$\Lambda(8)$	0.068	0.311	0.000	0.905	$\xi(8)$	0.009	0.95
$\Lambda(9)$	0.002	0.800	–1.640	0.000	$\xi(9)$	0.002	0.00
$\Lambda(10)$	0.087	0.092	0.203	0.026	$\xi(10)$	–0.001	0.04
$\lambda_1(1)$	–0.066	0.000	–0.427	0.000			
$\lambda_1(2)$	–0.034	0.000	–0.030	0.000			
$\lambda_1(3)$	2.725	0.000	0.045	1.000			
$\lambda_1(4)$	–1.025	0.000	0.055	1.000			
$\lambda_1(5)$	0.000	0.923	–0.001	0.723			
$\lambda_1(6)$	0.000	0.798	–0.002	0.408			
$\lambda_1(7)$	0.084	0.000	–0.012	0.000			
$\lambda_1(8)$	–0.015	1.000	0.000	0.884			
$\lambda_1(9)$	0.000	0.867	0.000	0.763			
$\lambda_1(10)$	0.659	0.000	–0.263	0.000			
λ_0	–9.882	0.000					
α	0.644	0.000					
$\widetilde{\alpha}^*$	0.083	0.000					

Estimated structural parameters B, λ_0, λ_1, Λ, λ_1^*, Λ^* are related to the ones of the initial state-space model by identities $C = \Lambda B$, $C^* = \Lambda^* B^*$, $c_1 = \lambda_1$, $c_1^* = \lambda_1^*$, $\zeta = C^* - C$, $\zeta_1 = c_1^* - c_1$, $\xi = \zeta_1 + \zeta\Phi$.

P-values are computed using the likelihood ratio principle extended to *m*-estimators (Hall, 2005). Specifically, let $\hat{K}$ be the value of the objective under unrestricted model and $\tilde{K}$ the value of the objective under restrictions. We deal with two kinds of restrictions: (i) a given parameter equal to zero (the case of parameters Λ, λ, λ_1, λ_0), or (ii) a function of parameters equal to zero (the case of parameters ξ, since they are linear combinations of the estimated parameters Λ, Λ^*,λ_1 and λ_1^*).

The likelihood ratio statistic is equal to $LR = -2T(\hat{K} - \tilde{K})$, where T is the number of observations. Asymptotically, under the null hypothesis that the tested restriction holds, the test statistic has an χ^2 distribution with degrees of freedom equal to the number of linearly independent restrictions.

the domestic and foreign perspective. Upon conducting the Kalman filter procedure, one obtains the estimates for structural parameters (cf. the note to Table 2) and subsequently – for the parameters Φ, C, C^*, c_0, c_1, c_1^*, $\{\gamma^j\}$, ξ and ζ appearing in (19)–(23).

The estimation algorithm was coded in MATLAB. Details of the used estimation technique are discussed in the Appendix. The estimation results for the state autoregression matrix Φ (cf. (19)) are given in Table 1 and for the other structural xparameters – in Table 2. Table 2 also features coefficients ξ (cf. (22)) that characterize the autarchic return on CZK cash conditioned on the previous period information.

5. Results and Conclusion

The immediate conclusion to be drawn from the estimation results is that the latent risk factor which is in the model responsible for both the EUR--CZK cross border order flow and the divergence of the actual from the autarchic exchange rate, is highly significant. In other words, there is usually a driving force behind the observed exchange rate movements that cannot be reduced to standard macroeconomic fundamentals.

Among the latter, there appears to exist a fairly prominent indicator of economic activity in both the euro area and the Czech economies with a significant relevance for the autarchic CZK/EUR exchange rate. In our setting it is represented by the industrial production growth in both economies. There is also a role for the expectations of future real activity, represented by the two business sentiment indicators. On the contrary, inflation variables in either economy have turned out to be of limited relevance (cf. coefficients $\xi(3) = -0.029$ and $\xi(8) = 0.009$ in Table 2). This may have to do with the fact that our sample only covers a stably low inflation period. Among the financial variables, the short and the long ends of the yield curve are only significant for the "peripheral" currency (Czech koruna), but not the central one (euro).

Another way of evaluating relative contributions of macro risk factors to explaining the exchange rate is to look at the degree of uncertainty generated in the one-period-ahead autarchic exchange rate by each of the observed macro variables. In particular, from (12) we know that the conditional expectation of the autarchy FX return is $E_t z_{t+1} = \xi u_t = \xi B^{-1} w_t$ (w is the macro variables vector). Define $\widetilde{\xi} = \xi B^{-1}$, so that $E_t z_{t+1} = \widetilde{\xi} w_t$. Thus it is possible to write

$$Var_t z_{t+1} = \sum_i \widetilde{\xi}_i \Big\{ Var_t w_i + \sum_{j \neq i} \widetilde{\xi}_j Cov_t(w_i, w_j) \Big\}$$

for the conditional variance of z. Accordingly, relative contributions to this conditional variance are given by

$$\frac{\widetilde{\xi}_i \Big\{ Var_t w_i + \sum_{j \neq i} \widetilde{\xi}_j Cov_t(w_i, w_j) \Big\}}{Var_t z_{t+1}}$$

Numerical results in per cent for the $\widetilde{\xi}$-values following from the estimated parameters ξ and B are given in the last column of Table 1. From it, we infer that the autarchic exchange rate *uncertainty*, as opposed to its *level*, is predominantly driven by the economic sentiment and the interest rate variables. The role of inflation variables is, again, modest.

Finally, we note that the defined framework for capturing the difference between the actual and the autarchic return on the peripheral currency from the center currency investor perspective allows one to give a reasonable explanation of the Czech koruna bubble that started in 2001 and continued late into 2002. As one sees in Figure 2, the first half of that period is characterized by a rapid growth of the autarchic koruna value (until the beginning of 2002; high positive autarchic return values), followed by a stabilization in the second half (autarchic return oscillating around zero). At the same time, the initial koruna appreciation was much lower, corresponding to the currency value only gradually catching up with the implicit autarchic level. In the second half of the period, the catch-up process finally lead to the closure of the FX order flow gap (actual koruna returns first rising above the autarchic ones, but eventually falling back to zero and even below). Thanks to this formal result, we are able to conclude that the Czech koruna behavior in the mentioned period was, indeed, distorted by a unidirectional flow of transactions not directly linked to macro fundamentals.[6]

The constructed international asset pricing model with an explicit FX order flow risk provides a method for estimating the divergence of the actual and the "balanced", i.e. inducing no excess currency demand, exchange rate. The outcomes allow one to assess the significance of the forex-specific risk factor in the investor's decision making. Accordingly, one becomes able to form and regularly update an opinion on the presence of free space for an active monetary authority role in the national currency market.

The presented analysis could be a useful supporting tool in the phase of the Czech Republic ERMII-entry and the selection of the euro central parity for the national currency. Supposing one establishes the significance of the cross-border FX order flow coefficient in the factor decomposition of the autarchic exchange rate, this could suggest the need to augment a purely macroeconomic view of the exchange rate behavior with the analysis of an additional latent component. Since the latter cannot be readily linked to the conventional set of macro fundamentals, one is confronted with a poorly classifiable source of nominal exchange rate volatility. Occasionally, this may be considered a legitimate reason for FX interventions. For those cases, our method might help the central bank to assess the extent of latent pressure it would have to face in the FX market.

More generally, a considerable shift in the estimated coefficients would indicate potential market pressure on the central parity, should it be fixed at the currently observed exchange-rate level. Namely, such a development would imply that the international investors' management of the country risks undergoes a revision that is not limited to the forex.

The cross-border FX order flow, reflected in the reported Czech bank spot FX transactions with non-residents, explains a large portion of the koruna/euro rate deviations from the uncovered asset return parity. This happens irrespective of whether specific episodes of pronounced one-sided flow have or do not have clearly identifiable fundamental reasons. Our analysis indicates that the range of risk factors influencing the observed FX order

[6] At the time, one was aware of the mass inflow of privatization revenues from abroad and, even more importantly, of the ongoing speculation about more inflow to follow, with the pending conversion of the euro proceeds into korunas by the government. However, this informal knowledge was not matched by any quantitative separation of an idiosyncratic privatization-induced FX inventory transfer on one side, from the overall international investor portfolio shift into Czech assets, based on a fundamental revision of their pricing paradigm, on the other (the latter would correspond to an upward shift in the autarchic koruna value). Our model indicates that this was not the case in 2001–2, but might very well be true for the second half of 2005. Indeed, for that period, one does not obtain any one-sided mismatch between the autarchic and the actual koruna values but rather a joint upward movement of both.

flow spans both standard macro fundamentals and idiosyncratic liquidity management-related ones that the central bank should keep track of. The relative importance of these factors has to be assessed before deciding the exact form and extent of the central bank presence in the market.

APPENDIX: Notes on Estimation Procedure

First, the models (19)–(24) are transformed into the following state-space model:

$$\widetilde{u}_{t+1} = \widetilde{\Phi}\widetilde{u}_t + \Phi_0 + \widetilde{\varepsilon}_{t+1} \tag{A2a}$$

$$\widetilde{y}_{t+1} = \widetilde{A} + \widetilde{B}\widetilde{u}_{t+1} + \widetilde{\nu}_{t+1} \tag{A2b}$$

where

$$\widetilde{u}_{t+1} = \begin{bmatrix} u_{t+1} \\ u_t \\ \varphi_{t+1} \end{bmatrix}, \quad \widetilde{\Phi} = \begin{bmatrix} \Phi & 0\ 0 \\ I & 0\ 0 \\ \frac{-1}{\alpha}\xi\Phi & 0\ 0 \end{bmatrix}, \quad \widetilde{\varepsilon}_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ \frac{-1}{\alpha}\xi\varepsilon_{t+1} \end{bmatrix},$$

$$\Phi_0 = \begin{bmatrix} 0 \\ 0 \\ \left[\frac{1}{\alpha}(\rho^0_{t+1} - \rho^i_{t+1}) - \zeta(\frac{\zeta}{2} + C)^T\right] \end{bmatrix}$$

$$\widetilde{y}^T_{t+1} = \lfloor w^T_{t+1} \quad y^{dT}_{t+1} \quad y^{fT}_{t+1} \quad \varphi^i_t, \ \varphi^{i*}_t \rfloor \qquad \widetilde{\nu}^T_{t+1} = \lfloor 0 \quad \nu^T_{t+1} \quad \nu^{*T}_{t+1} \quad \nu^\varphi_t \quad \nu^{*\varphi}_t \rfloor$$

$\widetilde{u}_{t+1}$ is the vector of current and lagged values of unobservable states (macroeconomic risks), and the state-transition matrix $\widetilde{\boldsymbol{\Phi}}$ and state-equation noise $\widetilde{\varepsilon}_{t+1}$ are partitioned accordingly. The measurement vector is composed of observed macroeconomic variables w_{t+1}, yields of assets observed from the domestic perspectives y^d_{t+1}, yields of assets observed from the foreign perspectives y^f_{t+1} and two proxies φ^i_t, φ^{i*}_t for the unobserved order flow. The covariance matrix of the state-equation disturbances $\widetilde{\varepsilon}_{t+1}$ is given by

$$\begin{bmatrix} I & 0 & \frac{-1}{\alpha}\xi^T \\ 0 & 0 & 0 \\ \frac{-1}{\alpha}\xi & 0 & \frac{1}{\alpha^2}\xi\xi^T \end{bmatrix}$$

As was explained in Section 3, both the theoretical model and the identification requirements impose a number of constraints on the matrices of the measurement equation. These matrices depend on parameters as follows:

$$\widetilde{A} = \begin{bmatrix} 0 \\ -c_0 - \|\gamma^1\|^2 / 2 \\ \vdots \\ -c_0 - \|\gamma^J\|^2 / 2 \\ -c_0^* - \|\gamma^1\|^2 / 2 \\ \vdots \\ -c_0^* - \|\gamma^J\|^2 / 2 \\ \widetilde{\alpha}_0 \\ \widetilde{\alpha}_0^* \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B & 0 & 0 \\ (\gamma^1 - C) & -(c_1 + \gamma^1 \Phi) & 0 \\ \vdots & \vdots & 0 \\ (\gamma^J - C) & -(c_1 + \gamma^J \Phi) & 0 \\ (\gamma^1 - C^*) & -(c_1^* + \gamma^1 \Phi) & 0 \\ \vdots & \vdots & 0 \\ (\gamma^J - C^*) & -(c_1^* + \gamma^J \Phi) & 0 \\ 0 & 0 & \widetilde{\alpha} \\ 0 & 0 & \widetilde{\alpha}^* \end{bmatrix}$$

Structural parameters of the model to be estimated are: Φ, B, λ_0, λ_1, Λ, λ^*_1, Λ^*, $\{\gamma^j\}$, α, $\widetilde{\alpha}$, $\widetilde{\alpha}^*$, $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_0^*$, with $c_0 = \lambda_0$, $c_1 = \lambda_1$, $C = \Lambda B$ (and similarly for c_0^*, c_1^*, C^*), $\zeta = C^* - C$, $\zeta_1 = c_1^* - c_1$, $\xi = \zeta_1 + \zeta\Phi$, $\lambda_0 = \lambda^*_0$. These relationships – along with assumptions on the covariance matrix of the measurement errors $\widetilde{\nu}_{t+1}$ – parameterize the statespace model. Note that it is impossible to separately identify all alphas.[7] Therefore we impose the restrictions $\widetilde{\alpha}_0 = \widetilde{\alpha}_0^* = 0$, $\widetilde{\alpha} = 1$.

Second, one must select an estimation method for the equation system (A2). Commonly, the estimation is based on Kalman filter recursion. The Kalman filter is used for the linear projection $\widetilde{u}^t_{t+1}$ of the unobserved state vector $\widetilde{u}_{t+1}$ on data $\{y_\tau\}^t_{\tau=1}$. In the case that the random noise in (A2) is i.i.d. Gaussian, then the Kalman-filter based projection is optimal with respect to the quadratic loss function, i.e., it holds that

$$E_t\left\{(\widetilde{y}_{t+1} - \widetilde{A} - \widetilde{B}\widetilde{u}^t_{t+1})^T(\widetilde{y}_{t+1} - \widetilde{A} - \widetilde{B}\widetilde{u}^t_{t+1}) - (\widetilde{y}_{t+1} - G\{\widetilde{y}_\tau\}^t_{\tau=1})^T(\widetilde{y}_{t+1} - G\{\widetilde{y}_\tau\}^t_{\tau=1})\right\} \leq 0 \quad \text{(A3)}$$

where E_t is the expectation operator with respect to the information available at time t, and G is any measurable, non-anticipating function of data. If random noises are not Gaussian, then the Kalman filter is optimal among linear forecasting rules, i.e., (A3) still holds for linear functional G. We denote Kalman-filter innovations as follows:

$$\varpi_{t+1} \equiv (\widetilde{y}_{t+1} - \widetilde{A} - \widetilde{B}\widetilde{u}^t_{t+1}) \quad \text{(A4)}$$

If the random noises $\widetilde{\varepsilon}_{t+1}$, $\widetilde{\nu}_{t+1}$ are iid Gaussian, then the innovations ϖ_{t+1} are iid Gaussian too, as well as the conditional distribution of state filtration given data. Therefore, the parameters of the state-space model can be estimated using the maximum likelihood approach, which is asymptotically the optimal strategy. In such a case, the likelihood estimation machinery can be used to estimate the asymptotic variance--covariance matrix and hence the likelihood-based trinity of statistical tests can be used to make inference about parameters. The negative of the log of the likelihood function is given by

$$-2\log L = n_y T \log(2\pi) \sum_{t=1}^{T} + \log(\det\Sigma_t) + \sum_{t=1}^{T} \varpi^T_{t+1}\Sigma^{-1}{}_t\varpi_{t+1} \quad \text{(A5)}$$

where T is the sample size, $\boldsymbol{\Sigma}$ is the variance-covariance matrix of innovations, and

[7] Indeed, consider the regression of the form $\varphi^i_t = \delta_0 + \delta_1\varphi^{i*}_t$, then the instrumental variable estimation based on the FX returns as instruments would yield $\hat{\delta}_{1,IV} \xrightarrow{p} \widetilde{\alpha}/\widetilde{\alpha}^*$ and $\hat{\delta}_{0,IV} \xrightarrow{p} (\widetilde{\alpha}/\widetilde{\alpha}^*)/ / \widetilde{\alpha}^* = \hat{\delta}_{1,IV} - 1$. Thus we cannot identify alphas separately. Similarly, intercepts in (24) are not identified; an OLS procedure for (24) would not allow the consistent estimation of anything useful because of measurement errors.

n_y is the dimension of the measurement equation (in our case $n_y = 20$ since we have 10 macro variables, $8 = 2 \cdot 4$ asset returns and two order flow observations).

If random noises do not obey an iid Gaussian distribution, the maximization of (A5) can still yield an asymptotically consistent estimator of parameters, although such an estimator will not be asymptotically optimal. The asymptotic consistency is guaranteed against a large set of alternatives, including distribution of the random noise with heavy tails, random volatility and so on. Hamilton (1994) provides a formula for the asymptotic variance-covariance matrix of estimates in such a situation.

An alternative approach of estimation is to form an m-estimator for (A4). Such an approach easily leads to the prediction error minimization criterion, i.e. the unknown parameters are set so as to minimize the sum of the sample prediction errors: $\sum_{t=1}^{T} \varpi_t^T W \varpi_t$ for a positive definite matrix $\boldsymbol{W}$. The idea is to use a two-stage m-estimator, with the second stagemaking use of the optimal weighting matrix consistently estimated in the first stage.

Numerical optimization of both estimation approaches has revealed that the faster convergence is achieved for the m-estimator. Thus, the tables below report results based on this method. The routine is programmed in MATLAB. The p-values are computed using the likelihood-ratio principle.

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SUMMARY

JEL Classification: F31, F41, G12, G15
Keywords: exchange rate; latent risk; order flow; pricing kernel; state space

Macroeconomic Factors and the Balanced Value of the Czech Koruna/Euro Exchange Rate

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The authors study the dependence of the Czech koruna's exchange rate to the euro on risk factors that cannot be reduced to standard macroeconomic fundamentals. For this purpose, they construct an international asset-pricing model in which the exchange rate is codetermined by a risk factor imperfectly correlated with other priced risks in the economy. The model embeds the standard no-arbitrage setup. It also contains an additional equation that links the autarchic currency price with the foreign-exchange order flow. In the state-space form, the unobserved variables that determine the dynamics of the asset markets, the autarchic exchange rate, and the FX order flow span a number of macroeconomic and latent risk factors. The model for the Czech koruna/euro exchange rate uses Kalman filter techniques. The results indicate the existence of a "non-fundamental" source of systematic divergence between the observed and the autarchic (i.e. fundamental) FX returns.