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Can We Still Benefit from International Diversification?
The Case of the Czech and German Stock Markets*

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Abstract
One of the findings of the recent literature is that the 2008 financial crisis caused a reduction in international diversification benefits. To fully understand the potential of diversification, we build an empirical model which combines generalized autoregressive score copula functions with high-frequency data and allows us to capture and forecast the conditional time-varying joint distribution of stock returns. Using this novel methodology and fresh data covering five years after the crisis, we compute the conditional diversification benefits to answer the question of whether it is still interesting for an international investor to diversify. As diversification tools, we consider the Czech PX and the German DAX broad stock indices, and we find that the diversification benefits strongly vary over the 2008–2013 crisis years.

1. Introduction

Proper quantification of the joint distribution allowing for time-varying dependence between assets is critical for asset pricing, portfolio allocation, and risk reduction. For a number of years, the finance literature has been studying the risk reduction benefit arising from international diversification. After the recent 2008 financial crisis, many researchers have documented a possible reduction in these benefits due to rising dependence between markets. However, the literature concentrating on the Central European markets has been limited, as it is widely believed that after the enlargement of the European Union these markets became integrated, with very limited opportunities for diversification.

In this paper, we revisit this line of research and study the possible benefits of diversification between the Czech PX and the German DAX stock market indices using data covering the five-year crisis period. While it is reasonable to believe that the Czech and German stock markets show a large degree of dependence due to the integration of the Czech Republic into the euro area and to the large dependence of the Czech economy on the German one, we aim to study whether the German and Czech stock indices can be considered for reducing the risk faced by an international investor.

A number of researchers have addressed the issue of the integration of Central and Eastern European (CEE) markets with the euro area. Voronkova (2004) docu-

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ments increasing stock integration between Central European (CE) markets and their mature counterparts in Europe, and finds lower diversification opportunities at the aggregate stock index level. Syriopoulos (2004, 2006) finds a long-run cointegrating relationship and hence limited diversification opportunities between CEE markets and Germany. Aslanidis and Savva (2011) confirm these findings using more recent data. Égert and Kočenda (2007) analyze the intraday interdependence of Western European and Central and Eastern European markets using a wide range of econometric techniques and find evidence of only short-term relationships among the CEE and Western European stock markets.

In a more recent study, Égert and Kočenda (2007) analyze the comovements of three developed and three emerging markets using a DCC-GARCH model on high-frequency data. They detect a small correlation between the developed and emerging markets. This finding is important for investors, as it allows them to diversify their portfolios by investing in the emerging markets. However, as the authors stress, this diversification opportunity may not be available more recently due to economic integration with Western Europe. A study by Hanousek and Kočenda (2011) uses high-frequency data to study foreign macroeconomic announcements and spillover effects on emerging CEE stock markets during 2004–2007. Among other findings, it is of interest that the Frankfurt stock market dominates the spillovers across the three emerging markets, while the reaction to the New York market is smaller. Syllignakis and Kouretas (2011) study the time-varying conditional correlations among the U.S., German, Russian, and CEE markets. The authors employ the DCC-GARCH model for the correlations and use weekly data spanning 1997––2009. They find a significant increase in the correlation between the U.S. and German markets and the CEE markets, especially during the 2007–2009 financial crisis.

In a recent study, Horváth and Petrovski (2013) analyze the comovements for Western Europe vis-à-vis Central and South Eastern Europe (SEE). The analysis is carried out on daily data for the period 2006–2011 using the bivariate BEKK-GARCH model. The authors find higher integration among CE countries and lower integration, with almost zero correlations, among SEE countries. In another study, Gjika and Horváth (2013) employ asymmetric DCC-GARCH to study the comovements in CE markets. Using daily data spanning 2001–2011, the authors find an increase in correlations after these countries joined the European Union, whereas asymmetric correlation effects were found only for the Hungary (BUX) and Poland (WIG) pair. In addition, a positive relation among the conditional correlations and conditional variances is confirmed, suggesting lower diversification in turbulent times.

A common feature of these studies is that they use cointegration, or multivariate GARCH, to study the dependence, and with some exceptions they use data from before the 2008 financial crisis. We contribute to the literature by using a very different approach proposed recently by Avdulaj and Baruník (2013) allowing us to model the time-varying joint distribution of stock market returns. Using the recently proposed time-varying copula methodology and utilizing high-frequency data, we build an empirical model which allows us to study the time-varying benefits of diversification. In addition, we contribute to the understanding of the relationship using recent data covering the crisis years. Using these fresh data and state of the art methodologies, we revisit the literature and uncover significant time variance in the benefits of diversification between the PX and DAX markets. This finding is particularly interesting, as the previous literature generally reports decreasing potential for diversification.

This paper is organized as follows. Section 2 introduces our empirical model, which is composed of the realized GARCH and generalized autoregressive score time-varying copulas. Section 3 introduces the data we use, while Section 4 discusses the in-sample and out-of-sample fits of all the model specifications and chooses the one which best describes the data. Finally, Section 5 tests the economic implications of our empirical model. We first evaluate the quantile forecasts, which are central to risk management, and then study the time-varying diversification benefits implied by our model. The last section concludes.

2. Dynamic Copula Realized GARCH Modeling Framework

Here we introduce the empirical model used to describe the dependence between the German DAX and Czech PX stock indices. Our modeling strategy utilizes high-frequency data to capture the dependence in the margins and recently proposed dynamic copulas to model the dynamic dependence. The final model is thus able to describe the conditional time-varying joint distribution of the returns.

The methodology is based on Sklar's (1959) theorem extended to conditional distributions by Patton (2006b). The extended Sklar's theorem allows us to decompose a conditional joint distribution into marginal distributions and a time-varying copula. Consider the bivariate stochastic process $\{\mathbf{X_t}\}_{t=1}^{T}$ with $\mathbf{X_t} = \left(X_{1t}, X_{2t}\right)'$, which has a conditional joint distribution $\mathbf{F_t}$ and conditional marginal distributions F_{1t} and F_{2t}. Then

$$\mathbf{X_t} \left| \mathcal{F}_{t-1} \sim \mathbf{F_t} = \mathbf{C_t}\left(F_{1t}, F_{2t}\right) \right. \quad (1)$$

where $\mathbf{C_t}$ is the time-varying conditional copula of $\mathbf{X_t}$ containing all information about the dependence between X_{1t} and X_{2t}, and $\mathcal{F}_{t-1}$ is the available information set, usually $\mathcal{F}_t = \sigma\left(\mathbf{X_t}, \mathbf{X_{t-1}}, \ldots\right)$. Due to Sklar's theorem, we are able to construct a dynamic joint distribution $\mathbf{F_t}$ by linking together any two marginal distributions F_{1t} and F_{2t} with any copula function.[1] Theoretically, there is limitless number of valid joint distribution functions that can be created by combining different copulas with different margins, making this approach very flexible.

2.1 Time-Varying Conditional Marginal Distribution with Realized Measures

The first step in building an empirical model based on copulas is to model the margins. Since the largest part of the dependence in financial time series is in their variance, the majority of researchers use the generalized autoregressive conditional heteroskedasticity (GARCH) approach of Bollerslev (1986) in this step.

We use the latest advances in the literature, which improve volatility modeling by adding a realized volatility measure to the GARCH model. This approach utilizes high-frequency data to help explain the latent volatility. In contrast the standard GARCH(1,1) model, where the conditional variance of the i-th asset,

[1] Note that the information set for the margins is the same as that for the copula conditional density.

$h_{it} = var\left(X_{it} \middle| \mathcal{F}_{t-1}\right)$, is dependent on its past values h_{it-1} and past values of X_{it-1}^2, Hansen et al. (2012) propose to utilize a realized volatility measure and make h_{it} dependent on the realized variance as well. In this work, we restrict ourselves to the simple log-linear specification of the so-called realized GARCH(1,1). For the general framework of realized GARCH(*p*,*q*) models we suggest that the reader consult Hansen et al. (2012). While it is important to model the conditional time-varying mean $E\left(X_{it} \middle| \mathcal{F}_{t-1}\right)$, we also include the standard autoregressive (AR) term in the final modeling strategy. As we will find later, an autoregressive term of order no larger than two is appropriate for the DAX and PX return series, so we restrict ourselves to specifying AR(2) with the log-linear RealGARCH(1,1) model as in Hansen et al. (2012)

$$X_{it} = \mu_i + \alpha_1 X_{it-1} + \alpha_2 X_{it-2} + \sqrt{h_{it}}\, z_{it} \qquad \text{for } i = 1,2 \tag{2}$$

$$\log h_{it} = \omega_i + \beta_i \log h_{it-1} + \gamma_i \log RV_{it-1} \tag{3}$$

$$\log RV_{it} = \psi_i + \phi_i \log h_{it} + \tau_i\left(z_{it}\right) + u_{it} \tag{4}$$

where μ_i is the constant mean, h_{it} is the conditional variance, which is latent, RV_{it} is the realized volatility measured from high-frequency data, $u_{it} \sim N\left(0, \sigma_{iu}^2\right)$, and $\tau_i\left(z_{it}\right) = \tau_{i1} z_{it} + \tau_{i2}\left(z_{it}^2 - 1\right)$ is the leverage function. For RV_{it}, we use the high-frequency data and compute it as the sum of the squared intraday returns (Andersen et al., 2003; Barndorff-Nielsen and Shephard, 2004). Innovations z_{it} are modeled by the flexible skewed *t* distribution of Hansen (1994). This distribution has two shape parameters, a skewness parameter $\lambda \in \left(-1, 1\right)$ controlling the degree of asymmetry, and a degree of freedom parameter $\nu \in (2, \infty]$ controlling the thickness of the tails. When $\lambda = 0$, the distribution reduces to the standard Student's *t* distribution, and when $\nu \to \infty$, it becomes a skewed normal distribution, while for $\nu \to \infty$ and $\lambda = 0$, it becomes $N(0,1)$.

Thus, after the time-varying dependence in the mean and volatility is modeled, we are left with innovations

$$\hat{z}_{it} = \frac{X_{it} - \hat{\mu}_i - \hat{\alpha}_1 X_{it-1} - \hat{\alpha}_2 X_{it-2}}{\sqrt{\hat{h}_{it}}} \tag{5}$$

$$\hat{z}_{it} \left| \mathcal{F}_{t-1} \right. \sim F_i(0,1) \qquad \text{for } i = 1,2 \tag{6}$$

which have a constant conditional distribution with zero mean and unit variance. Then the conditional copula of $\mathbf{X_t} \left| \mathcal{F}_{t-1} \right.$ is equal to the conditional distribution of $\mathbf{U_t} \left| \mathcal{F}_{t-1} \right.$:[2]

[2] Since the probability integral transform is invertible, the copula function also describes the dependence of the returns $\mathbf{X_t}|F_{t-1}$.

$$\mathbf{U_t} \left| \mathcal{F}_{t-1} \sim \mathbf{C_t} \left(\gamma_0 \right) \right. \tag{7}$$

with γ being copula parameters and $\mathbf{U_t} = \left[U_{1t}, U_{2t} \right]'$ the conditional probability integral transform

$$U_{it} = F_i \left(\hat{z}_{it}; \phi_{i,0} \right) \qquad \text{for } i = 1,2 \tag{8}$$

2.2 Dynamic Copulas: "GAS" Dynamics in Parameters

The notion of time-varying copula models was introduced by Patton (2006b). In the subsequent literature, Lee and Long (2009) develop a model where the multivariate GARCH is extended by copula functions to capture the remaining dependence. Recently, Hafner and Manner (2012) and Manner and Segers (2011) propose stochastic copula models, which allow the parameters to evolve as a latent time series. Another possibility is offered by ARCH-type models for volatility (Engle, 2002) and related models for copulas (Patton, 2006b; Creal et al., 2013), which allow the parameters to be some function of lagged observables. An advantage of the second approach is that it avoids the need to "integrate out" the innovation terms driving the latent time series processes.

With time-varying copula models, the driving dynamics of the model are of crucial importance. For our empirical model, we therefore adopt the generalized autoregressive score (GAS) model of Creal et al. (2013), which specifies the time-varying copula parameter (δ_t) as a function of the lagged copula parameter and a forcing variable that is related to the standardized score of the copula log-likelihood.[3] This type of dynamics reduces the one-step-ahead prediction error at the current observation given the current parameter values of the copula function. Consider a copula with time-varying parameters:

$$\mathbf{U_t} \left| \mathcal{F}_{t-1} \sim \mathbf{C_t} \left(\delta_t \left(\gamma \right) \right) \right. \tag{9}$$

Often, a copula parameter is required to fall within a specific range. For example, the correlation for a normal or Student's t copula is required to fall between the values of -1 and 1. To ensure this, Creal et al. (2013) suggest transforming the copula parameter using an increasing invertible function $h(\cdot)$ (e.g., logarithmic, logistic, etc.) to the parameter

$$\kappa_t = h\left(\delta_t \right) \Leftrightarrow \delta_t = h^{-1}\left(\kappa_t \right) \tag{10}$$

For a copula with a transformed time-varying parameter κ_t, the GAS(1,1) model is specified as

$$\kappa_t = \omega + \beta \kappa_{t-1} + \alpha I_t^{-1/2} \mathbf{s}_{t-1} \tag{11}$$

$$\mathbf{s_{t-1}} \equiv \frac{\partial \log \mathbf{c}\left(\mathbf{u_{t-1}}; \delta_{t-1} \right)}{\partial \delta_{t-1}} \tag{12}$$

$$I_t \equiv E_{t-1}\left[\mathbf{s_{t-1}} \mathbf{s'_{t-1}} \right] = I\left(\delta_t \right) \tag{13}$$

[3] Harvey (2013) and Harvey and Sucarrat (2012) propose a similar method for modeling time-varying parameters, which they call a dynamic conditional score model.

While this specification for the time-varying parameters is arbitrary, Creal et al. (2013) motivate it in such a way that the model nests a variety of popular approaches from conditional variance models to trade durations and counts models. Also, the recursion is similar to numerical optimization algorithms such as the Gauss-Newton algorithm.

Now we have specified the model, the last step is to choose the copula function used in the application. The time-varying copulas we use in this work are the rotated Gumbel, normal, and Student's *t*. In addition, we use constant copulas for comparison. To save space, we do not provide the functional forms of the copula functions used in this paper. These can be found in Patton (2006b).

2.3 Estimation Strategy

The final dynamic copula realized GARCH model defines a dynamic parametric model for the joint distribution. The joint likelihood is defined as

$$\mathcal{L}(\boldsymbol{\theta}) \equiv \sum_{t=1}^{T} \log \mathbf{f_t}\left(\mathbf{X_t};\boldsymbol{\theta}\right) = \sum_{t=1}^{T} \log f_{1t}\left(X_{1t};\theta_1\right) + \sum_{t=1}^{T} \log f_{2t}\left(X_{2t};\theta_2\right) \tag{14}$$

$$+\sum_{t=1}^{T} \log \mathbf{c_t}\left(F_{1t}\left(X_{1t};\theta_1\right), F_{2t}\left(X_{2t;}\theta_2\right);\theta_c\right) \tag{15}$$

where $\boldsymbol{\theta} = (\phi', \gamma')'$ is a vector of all the parameters to be estimated, including the parameters of the marginal distributions ϕ and the parameters of the copula γ . The parameters are estimated using a two-step estimation procedure, generally known as multi-stage maximum likelihood (MSML) estimation, first estimating the marginal distributions and then estimating the copula model conditional on the estimated marginal distribution parameters. While this greatly simplifies the estimation, performing inference on the resulting copula parameter estimates is more difficult than usual, as the estimation error from the marginal distribution must be taken into account. As a result, MSMLE is asymptotically less efficient than one-stage MLE. However, as discussed by Patton (2006a), this loss is not great in many cases. Moreover, the bootstrap methodology can be used for statistical inference.

2.3.1 Semiparametric Models

One of the appealing alternatives to a fully parametric model is to estimate the univariate distribution nonparametrically, for example by using the empirical distribution function. The combination of a nonparametric model for the marginal distribution and a parametric model for the copula results in a semiparametric copula model, which we use for comparison against its fully parametric counterpart. Forecasts based on semiparametric estimation where the nonparametric marginal distribution is combined with the parametric copula function are not common in the economic literature, so it is interesting to compare them in our modeling strategy. For the margins of the semiparametric models, we use the nonparametric empirical distribution F_i introduced by Genest et al. (1995),[4] which consists of modeling the marginal distributions by the (rescaled) empirical distribution.

[4] The asymptotic properties of this estimator can be found in Chen and Fan (2006).

$$\hat{F}_i(z) = \frac{1}{T+1}\sum_{t=1}^{T}\mathbf{1}\{\hat{z}_{it} \le z\} \tag{16}$$

In this case, the parameter estimation is conducted by maximizing the likelihood

$$\mathcal{L}(\gamma) \equiv \sum_{t=1}^{T}\log \mathbf{c_t}\left(\hat{U}_{1t};\hat{U}_{2t};\gamma\right) \tag{17}$$

As can be seen, the likelihood decreases in the estimation of the copula parameters only. However, we should note that performing inference on the parameters is more difficult than usual, hence we rely on bootstrap inference as advocated in Patton (2006a).

2.4 Copula Selection

An important issue when working with copulas is the selection of the best copula from the pool. Several methods and tests have been proposed for the selection procedure. The methods proposed by Durrleman et al. (2000) are based on the distance from the empirical copula. Chen and Fan (2005) propose the use of a pseudo-likelihood ratio test for selecting semiparametric multivariate copula models.[5] A test for conditional predictive ability (CPA) is proposed by Giacomini and White (2006). This is a robust test which allows both unconditional and conditional objectives to be accommodated. Recently, Diks et al. (2010) have proposed a test for comparing the predictive ability of competing copulas. This test is based on the Kullback-Leibler information criterion (KLIC) and its statistic is a special case of the unconditional version of Giacomini and White (2006).

As our main aim is to use the model for forecasting, the out-of-sample performance of models will be tested by CPA, which considers the forecast performance of two competing models conditional on their estimated parameters to be equal under the null hypothesis

$$H_0 : E\left[\hat{\mathbf{L}}\right] = 0 \tag{18}$$

$$H_{A1} : E\left[\hat{\mathbf{L}}\right] > 0 \text{ and } H_{A2} : E\left[\hat{\mathbf{L}}\right] < 0 \tag{19}$$

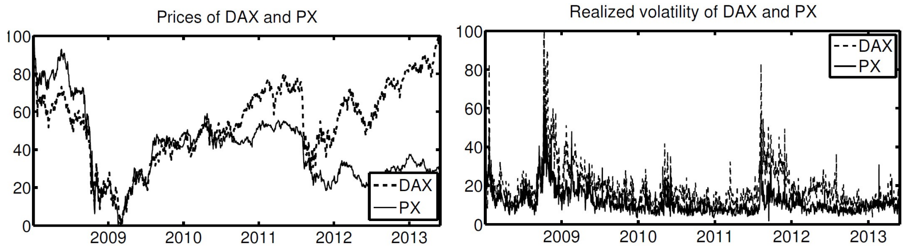
where $\hat{\mathbf{L}} = \log \mathbf{c_1}\left(\hat{\mathbf{U}},\hat{\gamma}_{1t}\right) - \log \mathbf{c_2}\left(\hat{\mathbf{U}},\hat{\gamma}_{2t}\right)$. Other advantages of this test are that it can be used for both nested and non-nested models and also for comparing parametric and semiparametric models. The asymptotic distribution of the test statistic is $N(0, 1)$ and we compute the asymptotic variance using HAC estimates to correct for possible serial correlation and heteroskedasticity in the differences in log-likelihoods.

3. Data Description

The data set consists of the five-minute prices of the Prague PX and German DAX stock indices over the period January 3, 2008–May 31, 2013, covering the re-

[5] Although some authors use AIC (or BIC) for choosing between two copula models, selection based on these indicators may hold only for the particular sample under consideration (due to their randomness) and not in general. Thus, proper statistical testing procedures are required (see Chen and Fan, 2005).

Figure 1 Normalized Prices and Annualized Realized Volatilities of the DAX and PX over the Sample Period Extending from January 3, 2008 until May 31, 2013



cent recession. We synchronize the data using time stamp matching and eliminate transactions executed on Saturdays, Sundays, holidays, December 24–26, and December 31–January 2 due to low activity on these days, which could lead to estimation bias. Hence, in our analysis we work with data for 1,349 days. For the empirical model, we need two time series, namely, daily returns and realized variance, to be able to estimate the realized GARCH model in the margins. For this, we obtain the daily returns as the sum of the logarithmic intraday returns, hence we work with open-close returns. The realized variance is computed as the sum of the squared five-minute intraday returns. *Figure 1* plots the evolution of PX and DAX prices together with their realized volatility. Note that the plot of prices is normalized so that we can compare the movements, and for the plot of realized volatility we use the daily volatility annualized according to the convention $100\times\sqrt{250\times RV_t}$. Strong time variance of the volatility can be noticed immediately for both the PX and DAX indices. The realized volatility of the DAX is larger on average than the volatility of the PX. Otherwise, the volatility has similar distributional properties for both indices.

4. Empirical Results

Before modeling the dependence structure between the PX and the DAX, we need to model their conditional marginal distributions. Considering general AR models up to five lags, we find AR(2) to best capture the time-varying dependence in the mean of the PX, while the DAX has a constant mean. These results are in line with previous research (Baruník, 2008). *Table 1* summarizes the realized GARCH(1,1) fit for both the PX and the DAX. In addition, the benchmark GARCH(1,1) model is fitted to the data for comparison. All the estimated parameters are significantly different from zero. By observing the partial log-likelihood $\mathcal{LL}_r$ as well as the information criteria, we can see that including realized measures in the GARCH model improves the fit significantly. This is crucial for copulas, as we need to specify the best possible model in the margins to make sure there is no univariate dependence left. If a misspecified model is used for the marginal distributions, the probability integral transforms will not be uniformly distributed and this will result in copula misspecification. For the estimated standardized residuals from the AR(2) realized GARCH(1,1), we consider both parametric and nonparametric distributions, as motivated earlier in the text.

Table 1 Parameter Estimates from AR(2) *log-linear* Realized GARCH(1,1) and Benchmark GARCH(1,1) with Innovations Distributed *skew-t* and Normal respectively
t-statistics Reported in Parentheses

	DAX		PX			DAX		PX	
	AR (2)					AR(2)			
c	0.0000	(0.08)	-0.0005	(-0.96)	c	0.0000	(0.08)	-0.0005	(-0.96)
α_1	-	-	0.0842	(3.10)	α_1	-	-	0.0842	(3.10)
α_2	-	-	-0.1069	(-3.94)	α_2	-	-	-0.1069	(-3.94)
	Realized GARCH(1,1)					GARCH(1,1)			
ω	0.2000	(7.64)	0.1794	(6.82)	κ	0.0113	(2.90)	0.0124	(3.28)
β	0.5746	(21.19)	0.6600	(23.31)	Φ	0.0852	(6.15)	0.1498	(7.43)
γ	0.4072	(14.22)	0.3399	(11.55)	Ψ	0.9022	(61.01)	0.8424	(45.39)
ξ	-0.5376	(-12.05)	-0.5834	(-12.13)		-	-	-	-
Φ	0.9655	(22.75)	0.8996	(23.46)		-	-	-	-
τ_1	-0.1691	(-13.26)	-0.1414	(-9.44)		-	-	-	-
τ_2	0.0717	(8.97)	0.0943	(10.23)		-	-	-	-
v	13.6919	(3.09)	7.3569	(5.27)		-	-	-	-
λ	-0.1161	(-3.39)	-0.0830	(-2.30)		-	-	-	-
$\mathcal{LL}_r$	-2461.00		-2545.41			-		-	
$\mathcal{LL}_r$	-1606.75		-1508.11		$\mathcal{LL}$	-1682.13		-1522.45	
AIC	3231.50		3034.23		AIC	3370.27		3050.90	
BIC	3278.37		3081.09		BIC	3385.89		3066.52	

4.1 Time-Varying Dependence between the DAX and the PX

Before specifying a functional form for the time-varying copula function, we test for the presence of time-varying dependence using a simple approach based on the ARCH LM test. The test statistic is computed from the OLS estimate of the covariance matrix, and the critical values are obtained using an *i.i.d.* bootstrap (for detailed information, consult Patton, 2012). Computing the test for the time-varying dependence between the DAX and the PX up to p = 10 lags, we find joint significance of all the coefficients. Thus, we can conclude that there is evidence against constant correlation for the DAX and the PX. Motivated by this finding, we estimate three time-varying copula functions, namely, the normal, rotated Gumbel and Student's *t*, using the GAS framework described in the methodology part. As a benchmark, we also estimate the constant copulas to be able to compare the time-varying models with the constant ones. As the semiparametric approach is empirically interesting and not often used in the literature, we use it for all the estimated models as well.

Table 2 shows the fit from all the models estimated. Starting with the constant copulas, all the parameters are significantly different from zero and the normal copula seems to describe the DAX and PX indices best according to the highest log-

Table 2 Constant and Time-Varying Copula Model Parameter Estimates with AR(2)-Realized GARCH(1,1) Model for Both Fully Parametric and Semiparametric Cases
Bootstrapped Standard Errors are Reported in Parentheses

		Paramatric			Semiparametric		
Constant copula		**Est.**	**Param**	**log $\mathcal{L}$**	**Est.**	**Param**	**log $\mathcal{L}$**
Normal	ρ	0.6042	(0.0188)	**305.90**	0.6053	(0.0157)	**307.83**
Clayton	κ	0.8596	(0.0591)	**221.98**	0.9258	(0.0560)	**232.08**
RGumb	κ	1.5819	(0.0385)	**265.24**	1.6130	(0.0323)	**272.99**
Student's *t*	ρ	0.5960	(0.0192)		0.6076	(0.0139)	
	v^{-1}	0.0100	(0.0224)	**305.53**	0.0100	(0.0042)	**307.43**
Sym. Joe-Clayton	τ^L	0.3667	(0.0295)		0.3991	(0.0318)	
	τ^U	0.3514	(0.0366)	**273.96**	0.3552	(0.0356)	**279.76**
"GAS" time-varying copula		**Est.**	**Param**	**log $\mathcal{L}$**	**Est.**	**Param**	**log $\mathcal{L}$**
$RGumb_{GAS}$	$\hat{\omega}$	-0.0466	(0.1245)		-0.0037	(0.1103)	
	$\hat{\alpha}$	0.0466	(0.0520)		0.0207	(0.0496)	
	$\hat{\beta}$	0.9139	(0.2137)	**266.69**	0.9927	(0.2491)	**275.95**
N_{GAS}	$\hat{\omega}$	0.0121	(0.2883)		0.0131	(0.3311)	
	$\hat{\alpha}$	0.0244	(0.0371)		0.0274	(0.0377)	
	$\hat{\beta}$	0.9911	(0.2064)	**312.28**	0.9907	(0.2198)	**314.54**
t_{GAS}	$\hat{\omega}$	0.1466	(0.2740)		0.1159	(0.2481)	
	$\hat{\alpha}$	0.0662	(0.0379)		0.0755	(0.0375)	
	$\hat{\beta}$	0.8936	(0.1941)		0.9188	(0.1720)	
	$\hat{v}^{-1}$	0.0115	(0.0069)	**311.41**	0.0120	(0.0092)	**313.43**

likelihood. The semiparametric specifications combining a nonparametric distribution in the margins with a parametric copula function bring further improvement in the log-likelihoods. Importantly, the time-varying specifications bring a large improvement in the log-likelihoods and confirm strong time-varying dependence between the DAX and PX indices. Due to the large number of degrees of freedom, the t_{GAS} copula in fact converges to the normal one N_{GAS}, and thus the time-varying normal copula again best describes the data.

This is an interesting finding, as it confirms that after one models properly for the dependence in the margins of the distribution, there is no asymmetry left and the PX-DAX bivariate distribution is standard normal. To study the goodness of fit for all the specified models, we use[6] the Kolmogorov-Smirnov (KS) and Cramer-von

[6] The results of the in-sample goodness of fit tests are available from the authors upon request. We omit them from the text to save space.

Mises (CvM) test statistics with p-values obtained from 1,000 simulations. None of the fully parametric models is rejected, while most of the semiparametric models are rejected with exception of constant Student's t, Sym. Joe-Clayton, and time-varying Student's t. These results suggest that fully parametric models with the realized GARCH and parametric distribution in the margins are all well-specified. The realized GARCH thus seems to model all the dependence in the margins very well, which is crucial for good specification of the model in the copula-based approach. Interestingly, the semiparametric models are rejected and are not specified well, except for a few mentioned cases. This is in line with the results of Patton (2012), who finds rejections in semiparametric specifications on U.S. index data. Still, both tests strongly support the realized GARCH time-varying GAS copulas for modeling the joint distribution between the DAX and the PX.

4.2 Out-of-Sample Comparison of the Proposed Models

While it is important to have a well-specified model which describes the data, most of the time we are interested in using the model for forecasting. We thus conduct an out-of-sample evaluation of the proposed models. For this, the sample is divided into two periods. The first, in-sample, period is used to obtain parameter estimates from all the models and spans the period January 3, 2008 to May 2, 2012. The second, out-of-sample, period is then used for forecast evaluation. Due to the highly computationally intensive estimation of the models, we restrict ourselves to a fixed window evaluation, where the models are estimated only once and all the forecasts are done using the parameters recovered from this fixed in-sample period. This makes it even harder for the models to perform well in the highly dynamic data.

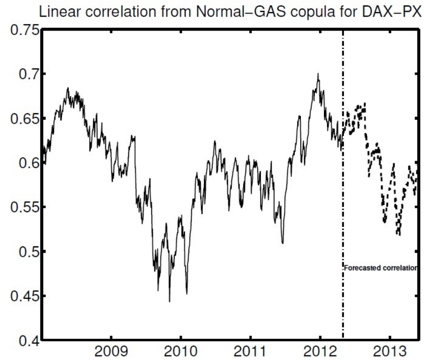
For the out-of-sample forecast evaluation,[7] we use the conditional predictive ability (CPA) test of Giacomini and White (2006). The time-varying copula models significantly outperform the constant copula models in the out-of-sample evaluation. This holds for both the parametric and semiparametric cases. Thus, time-varying copulas have much stronger support for forecasting the dynamic distribution of the DAX and the PX. When comparing the different time-varying copula functions, the test is not so conclusive. While the Student's t and normal copulas statistically outperform the rotated Gumbel copula, the forecasts from the Student's t copula cannot be statistically distinguished from the normal copula. The time-varying normal and Student's t copulas are thus the best performers in the forecasting exercise. Finally, forecasts from parametric models statistically outperform those from semiparametric ones.

Thus, the out-of-sample results confirm the in-sample ones, which is a good sign of proper model fit. The joint distribution of the PX and DAX indices is best modeled with the AR(2)-realized GARCH(1,1) time-varying normal copula model.

Having correctly specified the empirical model capturing the dynamic joint distribution between the DAX and the PX, we can proceed to studying the pair. *Figure 2* plots the time-varying correlations implied by our model with the normal GAS copula. The dependence is generally strong, and also has a strong time-varying

[7] The results of the out-of-sample forecast evaluation are available from the authors upon request. We omit them from the text to save space and to avoid distracting the reader from the main results.

Figure 2 Linear Correlation from the Time-Varying Normal GAS Copula.
The vertical dashed line separates the in-sample part from the out-of-sample (forecasted) part.



nature during the period studied. During the last quarter of 2008, when stock markets were declining due to the collapse of Lehman Brothers, the correlation between the PX and DAX markets rose to nearly 0.7. The following year it dropped to 0.45, and in 2010 it rose to 0.7 again.

This result has serious implications for investors, as it suggests that diversification possibilities have been rapidly changing during the financial crisis over the past few years. We are going to apply the results and study the possible economic benefits of the modeling strategy.

5. Economic Implications: Time-Varying Diversification Benefits and VaR

While it is important to have statistically correct fits, or even good out-of-sample forecasts, the crucial question is whether this translates to economic benefits. Here, we test our proposed methodology for its economic implications. First, we quantify the risk of an equally weighted portfolio composed of the DAX and PX, and second, we study the benefits of diversification to see how the strongly varying correlation affects them. This is of interest mainly to international investors considering including the Czech PX index and the German DAX index in their portfolios.

5.1 Quantile Forecasts

Quantile forecasts are central to risk management decisions due to widespread value-at-risk (VaR) measurement. VaR is defined as the maximum expected loss which may be incurred by a portfolio over some horizon with a given probability. Let q_t^α denote the α quantile of a distribution. The VaR of a given portfolio at time t is then simply

$$q_t^\alpha \equiv F_t^{-1}(\alpha) \qquad \text{for}\, \alpha \in (0,1) \tag{20}$$

Thus, the choice of the distribution is crucial to the VaR calculation. For example, assuming a normal distribution may lead to underestimation of the VaR. Our objective is to estimate the one-day-ahead VaR of an equally weighted portfolio composed of DAX and PX returns as $Y_t = 0.5X_{1t} + 0.5X_{2t}$, which has a conditional time-varying joint distribution F_t. In the previous analysis, we found that the realized GARCH model with the time-varying normal GAS copula fits and forecasts the data well, so we use it in VaR forecasts to see whether it also correctly forecasts the joint distribution. As there is no analytical formula which can be used for this, we rely on the Monte Carlo approach, where we simply simulate the future conditional joint distribution from the estimated models.

While quantile forecasts can be readily evaluated by comparing their actual (estimated) coverage $\hat{C}_\alpha = 1/n\sum_{n=1}^{T} 1\left(y_{t,t+1} < \hat{q}_{t,t+1}^{\alpha}\right)$ with their nominal coverage rate $C_\alpha = E\left[1\left(y_{t,t+1} < q_{t,t+1}^{\alpha}\right)\right]$, this approach is unconditional and does not capture the possible dependence in the coverage rates. A number of approaches have been proposed for testing the appropriateness of quantiles conditionally (for the best discussion, see Berkowitz et al., 2011). In our out-of-sample VaR testing, we use an approach originally proposed by Engle and Manganelli (2004), who use the n-th order autoregression $I_t = \omega + \sum_{k=1}^{n}\beta_{1k} I_{t-k} + \sum_{k=1}^{n}\beta_{2k} q_{t-k+1}^{\alpha} + u_t$, where I_{t+1} is 1 if $y_{t+1} < q_t^{\alpha}$ and zero otherwise. While the hit sequence I_t is a binary sequence, u_t is assumed to follow a logistic distribution and we can estimate it as a simple logit model and test whether $Pr\left(I_t = 1\right) = q_t^{\alpha}$. To obtain the p-values, we rely on simulations as suggested by Berkowitz et al. (2011), and we refer to this test as the DQ test in the results.

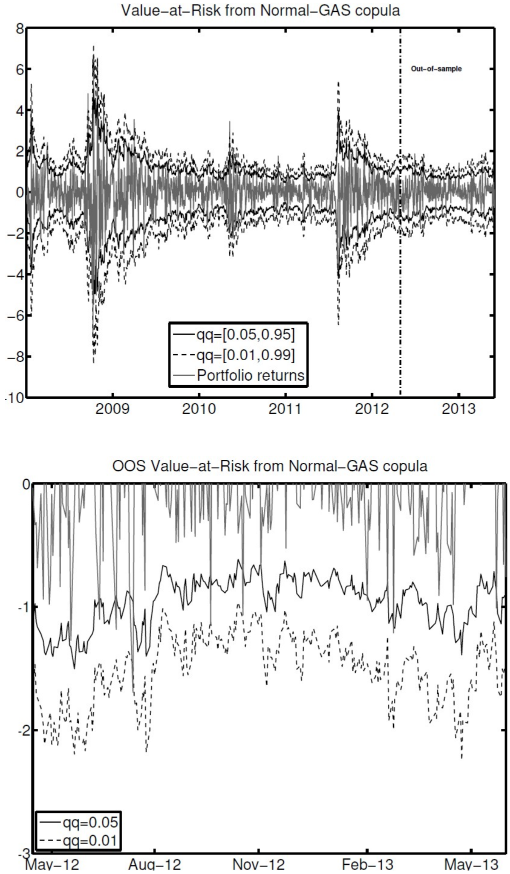
Moreover, we evaluate the accuracy of the VaR forecasts statistically by defining the expected loss of a VaR forecast made by a forecaster m as

$$L_{\alpha,m} = E\left[\alpha - 1\left(y_{t,t+1} < q_{t,t+1}^{\alpha,m}\right)\right]\left[y_{t,t+1} - q_{t,t+1}^{\alpha,m}\right] \tag{21}$$

which was proposed by Giacomini and Komunjer (2005). Differences in the values of $L_{\alpha,m}$ can then be tested using the Diebold and Mariano (2002) approach, where we test the null hypothesis that the loss function of a benchmark forecaster is the same as the loss function of the tested forecaster m, under the alternative that the benchmark forecaster is more accurate than the competing one.

Table 3 reports the out-of-sample VaR evaluation of all the models, and *Figure 3* illustrates the 1% and 5% estimated quantiles of the portfolio. We can see that all the time-varying models are well specified and the conditional quantile forecasts from them are not rejected by the DQ test. For the statistical testing, we use the time-varying normal copula as the benchmark forecaster and test all the other models against it. When looking at the loss functions $\hat{L}_{\alpha,m}$, we can see that norm of the quantiles implied by the different models can be distinguished from each other statistically, except for the 1% quantile. This is mainly because the Student's t copula has a large number of degrees of freedom, basically converging to the normal one. Thus, overall, AR(2)-realized GARCH(1,1) with time-varying copula models are able to describe and forecast the quantiles of the PX-DAX distribution very well.

Figure 3 Value-at-Risk Implied by Realized-GARCH Time-Varying Normal GAS Copulas. The Portfolio Consists of an Equal Weight of DAX and PX and the Estimation Is Made for the 1% and 5% Quantiles.



5.2 Time-Varying Diversification Benefits

When the dependence of the assets is changing strongly over time, it necessarily translates to changing diversification benefits as well. Unlike VaR, the expected shortfall satisfies the sub-additivity property and is a coherent measure of risk. Motivated by these properties, Christoffersen et al. (2012) propose a measure capturing the dynamics in diversification benefits based on the expected shortfall. The conditional diversification benefit (CDB) for a given probability level α is defined as

$$CDB_t^{\alpha} = \frac{\overline{ES}_t^{\alpha} - ES_t^{\alpha}}{\overline{ES}_t^{\alpha} - \underline{ES}_t^{\alpha}} \tag{22}$$

where ES^{α} is the expected shortfall of the portfolio at hand,

$$ES_t^{\alpha} = E\left[Y_t \middle| \mathcal{F}_{t-1}, Y_t \leq F_t^{-1}(\alpha) \right] \quad \text{for} \, \alpha \in (0,1) \tag{23}$$

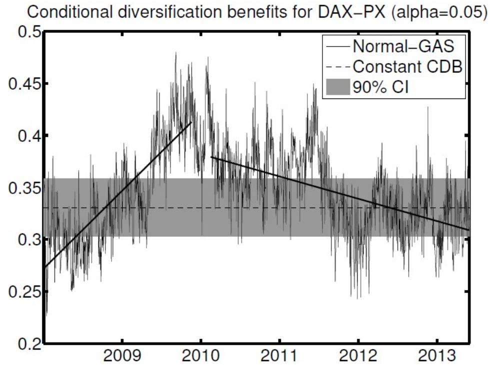
Table 3 Out-of-Sample VaR Evaluation. Empirical Quantile $\hat{C}_\alpha$, estimated Giacomini and Komunjer (2005) $\hat{L}$, logit DQ statistics and its 1000× simulated *p*-val are reported. $\hat{L}$ is moreover tested with (Diebold-Mariano statistics with Newey-West estimator for variance. All models are compared to N_{GAS}, while Models with Significantly Less Accurate Forecasts at 95% level are reported in bold.

	Parametric						Semiparametric					
	1%	5%	10%	90%	95%	99%	1%	5%	10%	90%	95%	99%
						$RGumb_{GAS}$						
$\hat{C}_\alpha$	0.004	0.033	0.074	0.915	0.948	0.989	0.004	0.033	0.070	0.922	0.948	0.989
$\hat{L}$	**0.017**	0.058	0.095	0.089	0.054	0.016	**0.017**	0.058	0.094	0.089	0.054	0.016
DQ	1.301	5.023	5.941	13.362	6.764	0.396	1.301	5.023	11.758	12.432	6.764	0.396
p-val	0.972	0.541	0.430	0.038	0.343	0.999	0.972	0.541	0.068	0.053	0.343	0.999
						t_{GAS}						
$\hat{C}_\alpha$	0.004	0.037	0.078	0.919	0.948	0.989	0.004	0.037	0.074	0.911	0.952	0.989
$\hat{L}$	0.017	0.058	0.094	0.088	0.054	0.016	**0.017**	0.057	0.094	0.089	0.054	0.016
DQ	1.301	4.663	5.781	12.735	6.764	0.396	1.301	4.663	5.941	14.299	5.779	0.396
p-val	0.972	0.588	0.448	0.047	0.343	0.999	0.972	0.588	0.430	0.026	0.448	0.999
						N_{GAS}						
$\hat{C}_\alpha$	0.007	0.037	0.070	0.922	0.948	0.989	0.011	0.033	0.074	0.926	0.952	0.989
$\hat{L}$	0.016	0.058	0.094	0.089	0.054	0.016	0.016	0.058	0.095	0.089	0.054	0.016
DQ	0.330	4.663	11.758	12.432	6.764	0.396	0.396	5.023	5.941	11.655	5.779	0.396
p-val	0.999	0.588	0.068	0.053	0.343	0.999	0.999	0.541	0.430	0.070	0.448	0.999

$\overline{ES}_t^\alpha$ is the upper bound on the expected portfolio shortfall, being the weighted average of the assets' individual expected shortfalls, and $\underline{ES}_t^\alpha$ is the lower bound on the expected shortfall, being the inverse cumulative distribution function for the portfolio. In other words, this lower bound corresponds to the case where the portfolio never loses more than its α distribution quantile. The measure is designed to stay within the $[0,1]$ interval and is increasing in the level of diversification benefits. When the CDB is equal to zero, there are literarily no benefits from diversification. When it equals one, the benefits from diversification are the highest possible.

Figure 4 plots the conditional diversification benefits for the PX and DAX portfolio implied by our empirical model for $\alpha = 5\%$. Similarly to the VaR case, as there is no closed-form solution to our empirical model available, we rely on the simulations for the CDB computation. Encouraged by the previous results, we compute the CDB for the AR(2) realized GARCH with the time-varying normal copula model. The analysis could be taken a step further by optimizing the portfolio weights for the highest diversification benefits. This is done in Christoffersen et al.

Figure 4 Conditional Diversification Benefits, CDB_t^{005}, Using the Time-Varying Normal Copula together with the Bootstrapped Confidence Band for the Constant Conditional Diversification Benefits



(2012), who basically find a very small increase, implying that an equally weighted portfolio is usually very close to optimal if the CDB is used. Please also note that here we do not exploit the full potential of dynamic asset allocation.

The diversification benefits vary greatly over time. From the beginning of the sample, the benefits of diversification between the DAX and the PX index rise gradually until the end of 2009, when they start to decline. The lowest values are at the beginning of 2012, while from this point until 2013 the benefits stay pretty low.

To support our results, we also report 90% bootstrapped confidence bands computed around a constant level of diversification benefits. Assuming the returns data are independently distributed over time with the same unconditional correlation as the PX and DAX pair, the bootstrap confidence level can be conveniently computed via simulations. We use 10,000 simulations and report the mean value together with the distribution of the constant conditional benefits in *Figure 4*. We can see that the time-varying nature of the conditional diversification benefits is statistically significant, as it departs from the simulated constant distribution.

Thus, contrary to the general expectation of no diversification benefits for investors considering the Czech PX index as a diversification tool for the German DAX due to the very high correlation between these two stocks, we find that the actual benefits vary strongly over time.

6. Conclusions

This work revisits the dependence between the Czech PX and German DAX stock markets with the aim of studying the opportunities of these two assets in portfolio management. Using an empirical model utilizing high-frequency data in time-varying copulas, we study the joint conditional distribution of the PX and DAX returns.

The final AR(2) realized GARCH(1,1) with a time-varying normal copula is able to capture the dynamics accurately, yielding precise quantile forecasts. Using the crisis data, we study the time-varying correlations between the PX and DAX returns. More importantly, we study how the time-varying dependence translates to the conditional diversification benefits. The main result is that the possible diversification benefits vary strongly over time, and hence even after the 2008 financial crisis, it may be economically interesting to use DAX and PX returns for risk diversification. This is an important finding, as it runs contrary to the belief that the crisis caused a reduction in international diversification benefits. The Czech and German economies are strongly interlinked as well, so one would expect the diversification benefits to disappear, especially after the Czech Republic joins the euro area.

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