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International Equity Portfolio Risk Modeling: The Case of the NIG Model and Ordinary Copula Functions*

Aleš KRESTA (ales.kresta@vsb.cz—*corresponding author*)
Tomáš TICHÝ (tomas.tichy@vsb.cz)
both authors: Department of Finance, Faculty of Economics, VŠB-TU Ostrava

Abstract
Financial risk modeling and management are very important and challenging tasks for financial institutions' quantitative units. Owing to the complex nature of portfolios, and given recent financial market developments, contemporary research is focused on tail modeling and/or dependency modeling. The main objective of this paper is to examine the potential contribution of Lévy-based subordinated models coupled by ordinary elliptical copula functions to the estimation of the distribution pattern of international equity portfolios. We observe that the subordinated NIG model coupled with the Student copula function, and in particular its combined estimation version, allows us to get very good estimates of portfolio risk measures.

1. Introduction

Financial institutions play a central role in the economic system, since their existence enables the efficient transfer of funds, liquidity, maturity, and also risk. Markets (and the world in general) are not frictionless—doing business is easier with a huge amount of funds, information usually cannot be obtained for free, financial instruments are difficult to understand, and so on. However, financial institutions have at their disposal a huge amount of funds and highly skilled staff, so that operations can be performed quickly and information obtained effectively.

Although well-managed financial institutions help to improve the condition of economic systems, badly run institutions can obviously do the opposite. To prevent financial institutions from failing and to increase confidence in the financial system, some sort of regulation and subsequent supervision is necessary. One of the most important elements of such regulation is to specify which models are appropriate for measuring risk exposures. In this paper we concentrate on market risk models and their ability to estimate portfolio exposure risk soundly.[1]

Market risk is commonly linked to a prespecified quantile of a probability distribution of a financial institution's portfolio. The most often used measure, VaR (Value at Risk), tells us the lowest return with a given confidence (α) or, alternatively, that with a given probability $q = 1 - \alpha$, the loss will be equal to or higher than the VaR. This measure of risk was introduced in JP Morgan at the end of the last

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[1] For further discussion of risk management by financial institutions, see, for example, Hull (2010) or Resti and Sironi (2007).

century as a simplifying but unifying proxy for a collection of many diverse risk measures for complex portfolios (see, for example, Jorion, 2007). The way VaR was defined was heavily criticized soon after it was introduced (see, for example, Artzner et al., 1999), so measurement of the conditional mean for a given probability $q = 1 - \alpha$, i.e., the average loss when things go wrong, has also been proposed. Such measures are denoted by various authors as CVaR (Conditional Value at Risk), ES (expected shortfall), and tail VaR.

The ability of (market) risk models to estimate risk exposures soundly is commonly assessed by the so-called backtesting procedure. It works as follows. At time t the risk (for example in terms of VaR) is estimated for time $t + 1$ (say, the next day). Later, it is compared with the true loss record. If the loss is higher than the estimate, that day is referred to as an *exception* and denoted by 1. Otherwise we record 0. The procedure is repeated over a given time length (usually several years). The sequence of 0s and 1s should fulfill some statistical assumptions. The simplest way is to compare the true number of exceptions to the assumptions about them (Kupiec, 1995). A review of some further techniques for statistical testing of exceptions can be found, for example, in Berkowitz et al. (2011).

Recently, several papers have been published dealing with the analysis of risk models via backtesting (mostly according to Kupiec's test). While, for example, Alexander and Sheedy (2008) assumed Gaussian/Student/GARCH/Empirical models for a simple position and Rank (2007) analyzed similar models of marginal distribution joined together by several copula functions, in Tichý (2010a,b) the performance of ordinary elliptical copula Lévy-driven models for FX rate sensitive portfolios was analyzed using various time spans to estimate the parameters. In this paper we extend the analysis to study an international equity index portfolio assuming a Lévy-type NIG model[2] for marginal distributions, i.e., two different sources of risk are present —FX rates and equity indices.

We proceed as follows. In Section 2 we provide some basic facts about the risk estimation methodology and model evaluation. Several interesting findings regarding VaR calculation and/or backtesting are reviewed in Section 3. Next, in Section 4, we define the model used for portfolio risk estimation—a multidimensional model in terms of Lévy marginals (the NIG model in particular) joined together by ordinary copula functions (either Gaussian or Student). Finally, the data set of four equity indices and the relevant FX rates with respect to CZK are described (Section 5) to provide a basis for the VaR backtesting process (Section 6). The most important findings are summed up in the *Conclusion*.

2. Risk Estimation and Model Evaluation

Supervisors' policies affect the risk management activities of financial institutions in two important ways. First, they specify eligible approaches to risk measurement, and second, they set risk limits, which should not be broken. For example, within the banking industry, since The Amendment 1992 (an amendment to

[2] The reasons are (i) that in Tichý (2010a) no important differences were documented between the performance of VG and NIG models, and (ii) that the distribution function of the NIG model is more suitable for approximations, which significantly decreases the computational time—see also Kresta et al. (2010) and Kresta (2011).

the original set of regulation rules and recommendations Basel I), financial institutions have been allowed to use internal models based on the VaR approach to quantify the market risk they are exposed to. The horizon over which the risk should be monitored was set at ten days with a pre-set confidence level of 99%. Although Basel I has recently been replaced by Basel II (which, in turn, is supposed to be replaced by Basel III), there is almost no modification concerning market risk measurement.

Similarly to the banking industry, insurance companies will also be required to follow a risk-based approach to minimum capital requirements.[3] This modification of Basel II is referred to as Solvency II and includes minimum (85%) and solvency (99.5%) capital requirements, both for a one-year horizon. The horizon in the case of insurance companies is so long because short-term variability should not matter—their financial market investments (for example high-rated bonds and equities) are intended as long-term holdings, in contrast to most investments of (investment) banks. However, in the banking industry, two portfolios are distinguished—the trading book (short-term holdings) and the banking book (long-term holdings, potentially to maturity). Obviously, the market risk capital requirement is calculated mostly only for the trading book and the horizon of ten days should be applied only to highly liquid instruments.

2.1 VaR Calculation

Assuming a random variable X (for example, the return on a portfolio), the VaR over time interval Δt with confidence α (i.e., with significance $q = 1 - \alpha$ can be obtained as follows:

$$VaR_X\left(\Delta t,\alpha\right) = -F_X^{-1}\left(q\right) \tag{1}$$

Here, $F_X^{-1}\left(q\right)$ denotes the inverse of the distribution function of random variable X for q (a quantile). If X follows a Gaussian distribution, we get:

$$VaR_X\left(\Delta t,\alpha\right) = -F_X^{-1}\left(q\right) = -\mu_X\left(\Delta t\right) - \sigma_X\left(\Delta t\right)F_N^{-1}\left(q\right) \tag{2}$$

where F_N^{-1} denotes the inverse of the distribution function of a standard Gaussian distribution, μ_X is the mean of X (i.e., the average portfolio return over Δt), and σ_X is its standard deviation.

However, it is rare for random variables, such as the returns on financial assets, to follow a Gaussian distribution. Usually, we have to select a distribution with some additional parameters so that we can either control the higher moments of the distribution or make the volatility stochastic. In that case, it can be inevitable to run a Monte Carlo simulation procedure to obtain the VaR as an estimate of $F_X^{-1}\left(q\right)$.

When originally introduced, VaR was calculated by the variance-covariance approach in its basic form using Gaussian innovations and GARCH-type models for volatility. Alternatively, the non-parametric approach of historical simulation can be

[3] This contrasts with the original Solvency concept, which was focused solely on the liabilities, i.e., each insurance company was required to be sufficiently solvent to meet all potential insurance claims from its provisions.

used, i.e., returns observed in the past can be re-used to estimate the quantile of the distribution. However, while the use of Gaussian innovations ignores the true pattern of the distribution, such as excess kurtosis, historical simulation is problematic when fat tails are to be estimated. This is why one might prefer the Monte Carlo simulation approach.

2.2 VaR Backtesting

Since for VaR models there is no measure of their ex ante goodness-of-fit, the quality of the model is usually judged by comparing actual losses with ex ante expectations over some horizon. Loosely speaking, applying the historical time series, i.e., the true evolution of market prices of financial instruments, the risk is estimated (ex ante) at time t for time $t + \Delta t$, where Δt is usually set to one business day, and compared with the true loss observed at time $t + \Delta t$ (ex post). This procedure is applied for a moving time window over the whole available data set. This is called backtesting.

In line with the banking supervision standards defined in Basel II, let us assume that the risk is estimated for a one-day horizon, $\Delta t = 1$. Denote the Value at Risk of a portfolio X estimated on day t for the next day $t + 1$ with a given confidence level α as $VaR_X(t, t+1; \alpha)$ and the true loss observed at time $t + 1$ with respect to the preceding day t as $L_X(t, t+1)$.

In the backtesting procedure on a given time series $t = \{1, 2, \ldots, T\}$, two situations can arise—either the loss is higher than estimated or it is lower than estimated (from the stochastic point of view, equality should not arise for continuous variables). The former case is denoted by 1 as an exception, and the latter one is denoted by 0:

$$I_X(t+1;\alpha) = \begin{cases} 1 \; if \;\; L_X(t,t+1) > VaR_X(t,t+1;\alpha) \\ 0 if \;\; L_X(t,t+1) \le VaR_X(t,t+1;\alpha) \end{cases} \tag{3}$$

On the sequence $\{I_X(t+1;\alpha)\}_{t=m}^{T-1}$, where m is the number of data (days) needed for the initial estimation, it can be tested whether the number of *ones* (exceptions) corresponds with the assumption, i.e., nq (with $n = T - 1 - m$), whether the estimate is valid either unconditionally or conditionally, whether bunching is present, and so on.

Since the distribution of exceptions over time should be identical and independent (i.i.d. Bernoulli variables), we can generally assume that with probability $1 - p$ the number of exceptions will be within the interval:

$$\left(nq - Z_{1-p/2}\sqrt{nq(1-q)}, nq + Z_{1-p/2}\sqrt{nq(1-q)}\right) \tag{4}$$

where Z_x denotes the value of the inverse of the distribution function of the standard normal distribution for x.[4]

[4] Here, p is the p-value at which we perform the test, and we use the standard normal distribution as a reasonable approximation of the binomial distribution for large n and small q.

The majority of the tests used in connection with the accuracy of VaR models use the LR (likelihood ratio) test procedure. Thus, the null hypothesis that the risk estimate is consistent with the assumption can be accepted only if the calculated LR does not exceed the *p*-critical value of the LR distribution.[5]

3. Previous Results on VaR Backtesting

Since VaR-based models have been widely accepted by financial market supervisors as the best available measure of market risk and in this way constitute a proxy for the minimum capital requirement, there are many papers that analyze the performance of various models in the risk estimation process.

Obviously, the first papers were focused mainly on ideal testing procedures. For example, Crnkovic and Drachman (1996) constructed a goodness-of-fit measure for an estimate of the entire profit and loss distribution. Applying a symmetric weighted function, they obtained an accuracy test for the tail-risk estimates. However, this testing procedure requires at least 1,000 observations in the case of an underlying symmetric distribution.

Around the same time, Kupiec (1995) analyzed the statistical properties of exceptions and their testing ability. He pointed out problems related to TUFF-based tests (time until first failure) and provided a limiting number of observations for a PF-based test (proportion of failures). While the former can be replaced by a duration-based test, where the time between particular failures is measured (see Christoffersen and Pelletier, 2004), the latter, also called the unconditional coverage test, should be accompanied by the conditional coverage test of Christoffersen (1998).

One of the first attempts to analyze the real behavior of VaR-based market risk models in depth was the study of internal bank models by Berkowitz and O'Brien (2002) using the data reported by particular banks to regulators in the US. It was concluded that the real models are rather conservative and do not adequately reflect daily volatility changes. The resulting higher and relatively stable risk estimates lead to higher capital requirements, although if an exception arises it can be very severe and potentially followed by some more exceptions. This line of research was followed, for example, by Pérignon and Smith (2010), who studied the level and quality of VaR disclosure in various countries. They documented an upward trend in the quality of information and found that the historical simulation approach, despite being the most popular method, showed relatively poor performance. They also questioned the usefulness of VaR for capital requirements from the point of view of regulatory bodies. This led Berkowitz et al. (2011) to reconsider the available internal data and suggest more powerful tests.

Concerning risk estimation for hypothetical positions rather than complex real portfolios, Pritsker (1997) studied accuracy versus computational costs. Later, again Pritsker (2006) analyzed the efficiency of the historical simulation approach to VaR estimation. He concluded that even two years of daily data may not be enough to obtain enough observations in the tails to estimate 99% VaR accurately.[6]

[5] Note that two types of errors can arise: a Type 1 error indicates that the true model will be rejected with probability *p*, while a Type 2 error provides the rate at which the false model is accepted.

[6] Recall that for internal purposes the risk can be estimated with much higher confidence than the Basel 99%.

The introduction of copula functions into finance in recent years allows one to combine various types of marginal distributions with a suitable dependency function (i.e., a copula). This has increased the power of parametric models. For example, Rank (see chapter 8 in Rank, 2007) studied Frank and Gumbel-Hougaard copula functions combined with Gaussian or Student marginals for risk estimation of zero-value (on average) simple FX rate portfolios and found that Student marginals with both copulas substantially outperform more standard approaches. More recently, Huang et al. (2009) combined non-parametric GARCH models for a marginal distribution with a large selection of copula functions for risk estimation of a portfolio of two stock indices. They suggested the GARCH-t-copula model as the optimal choice. Finally, Ignatieva and Platen (2010) suggested that preference should be given to the Student copula with Student marginals over alternative symmetric generalized hyperbolic distributions on the basis of an analysis carried out for a portfolio of stock indices based on the Heath-Platen benchmark approach.

The methodology we present in this paper differs from the previous research by combining a Lévy-type subordinated model (normal inverse Gaussian) with symmetric ordinary copula functions. We therefore allow asymmetry in the marginal distribution of particular risk factors. Moreover, we create a portfolio of stock indices denominated in different currencies so that two types of risk arise—FX rate risk and equity index risk. Finally, we compare the performance of particular models at various significance levels.

4. Methodology

The traditional approach of portfolio risk modeling via parametric models was based on the assumption of a joint—preferably multidimensional Gaussian—distribution. Although the assumption of normality can be accepted for huge and deeply diversified portfolios, i.e., across different markets, modeling the risk of equity or FX rate portfolios often requires a model that also takes into account the higher moments of the distribution and potentially also special features of the dependency between particular risk sources. It can therefore be fruitful to separate the modeling of particular risk factors (marginal distributions) and the dependency between them. Hence, in the following subsections we describe the two steps separately.

4.1 Marginal Distribution

A major challenge of financial model building is to fit extreme movements in market returns. It is a matter of fact that returns in financial markets are neither symmetrically distributed nor without sharp peaks (or heavy tails) over time, which is in contradiction to the assumption of Gaussian distribution. A very feasible way to fit both skewness (asymmetry) and kurtosis (heavy tails or high peaks) is to apply the subordinated Lévy model, a rather non-standard version of the Lévy model defined as time-changed Brownian motion, which dates back to Mandelbrot and Taylor (1967), Clark (1973), or even Bochner (1949).[7]

[7] The first focus on Lévy models with jumps dates back to the 1930s. The most recent and complete monographs on the theory and/or application of Lévy models include Applebaum (2004), Cont and Tankov (2004), and Barndorff-Nielsen et al. (2001).

Generally, a Lévy process is a stochastic process which is zero at origin. Its path in time is right-continuous with left limits, and its main property is that it is of independent and stationary increments. Another common feature is so-called stochastic continuity. Moreover, the related probability distribution must be infinitely divisible. For a given infinitely divisible distribution, we can define a triplet of Lévy characteristics:

$$\left\{\gamma, \sigma^2, \nu(\mathrm{d}x)\right\}$$

The first two define the drift (the deterministic part) and the diffusion of the process. The third is a Lévy measure. If it can be formulated as $\nu(dx) = u(x)dx$, it is a Lévy density. It is similar to the probability density, with the difference that it need not be integrable and zero at origin and is inevitable when Lévy copulas are to be defined. Moreover, the triplet above allows us to construct the characteristic equation of all Lévy processes, the so-called Lévy-Khintchine formula:

$$\Phi(u) = \mathrm{i}\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} \left(\exp(\mathrm{i}ux) - 1 - \mathrm{i}uxI_{|x|<1}\right)\nu(\mathrm{d}x) \tag{5}$$

Let us define a stochastic process $\mathcal{Z}(t;\mu,\sigma)$, which is a Wiener process (standard Brownian motion). As long as $\mu = 0$ and $\sigma = 1$, its increment within an infinitesimal time length dt can be expressed as:

$$\mathrm{d}\mathcal{Z} = \varepsilon\sqrt{dt}, \quad \varepsilon \in \mathcal{N}[0,1] \tag{6}$$

where $\mathcal{N}[0,1]$ denotes a Gaussian distribution with zero mean and unit variance. Then, a subordinated Lévy model can be defined as Brownian motion[8] driven by another Lévy process $\ell(t)$ with unit mean and positive variance κ. The only restriction for such a driving process is that it is non-decreasing over a given interval and has bounded variation.

Hence, we replace standard time t in Brownian motion X,

$$X(t;\theta,\vartheta) = \theta dt + \vartheta\mathcal{Z}(t) \tag{7}$$

by its function $\ell(t)$ as follows:

$$X\left(\ell(t);\theta,\vartheta\right) = \theta\ell(t) + \vartheta\mathcal{Z}\left(\ell(t)\right) = \theta\ell(t) + \vartheta\varepsilon\sqrt{\ell(t)} \tag{8}$$

Nevertheless, in order to model market returns it is important to fit the long-term return μ. Thus, we have to rewrite (8) as follows:

$$\mu t + \theta\left(\ell(t) - t\right) + \vartheta\varepsilon\sqrt{\ell(t)}, \tag{9}$$

so that the expectation will really be given by μ (the mean of $\ell(t)$ is t, while ε has zero expectation).

Due to their simplicity (tempered stable subordinators with a known density function in the closed form), the most suitable models seem to be either the variance

[8] For our purposes a Brownian motion is a Wiener process without any premise on μ and σ.

Table 1 Comparison of Selected Models

Model	Skewness	Kurtosis	Intrinsic time
Brownian motion	0	0	equivalent to *t*
variance gamma model $VG(\ell(t);\theta,\delta)$	nonzero if $\theta \neq 0$ its sign is determined by the sign of θ	presumably excess	$\ell(t) \in G[a,b]$
normal inverse Gaussian model $NIG(\ell(t);\theta,\delta)$	nonzero if $\theta \neq 0$ its sign is determined by the sign of θ	presumably excess	$\ell(t) \in IG[a,b]$

gamma model (where the overall process is driven by a gamma process from a gamma distribution with shape a and scale b depending solely on variance κ, $G[a,b]$) or the normal inverse Gaussian model (where the subordinator is given by an inverse Gaussian process based on the inverse Gaussian distribution, $IG[a,b]$) – see *Table 1* for a comparison.[9]

4.2 Joint Distribution

A useful tool for dependency modeling is the copula function,[10] i.e., the projection of the dependency between particular distribution functions into $[0,1]$,

$$\mathcal{C}:[0,1]^n \to [0,1] \text{ on } \mathbb{R}^n, \ n \in \{2,3,\ldots\} \tag{10}$$

Actually, any copula function can be regarded as a multidimensional distribution function with marginals in the form of a standardized uniform distribution.

For simplicity, assume two potentially dependent random variables with marginal distribution functions F_X, F_Y and a joint distribution function $F_{X,Y}$. Then, following Sklar's theorem:

$$F_{X,Y}(x,y) = \mathcal{C}\left(F_X(x), F_Y(y)\right) \tag{11}$$

If both F_X and F_Y are continuous, the copula function $\mathcal{C}$ is unique. Sklar's theorem also implies an inverse relation,

$$\mathcal{C}(u,v) = F_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right) \tag{12}$$

Formulation (12) above should be understood such that the joint distribution function gives us two distinct pieces of information: (i) the marginal distributions of the random variables, and (ii) the dependency function of the distributions. Hence, while the former is given by $F_X(x)$ and $F_Y(y)$, the copula function specifies

[9] The variance gamma model is basically due to Madan and Seneta (1990), while the normal inverse Gaussian model is due to Barndorff-Nielsen (1995) and (1998). Note also that several generalizations and extensions exist—see the monographs referred to above.

[10] In this paper, we restrict ourselves to ordinary copula functions. The basic reference for the theory of copula functions is Nelsen (2006), while Rank (2007) and Cherubini et al. (2004) focus mainly on the application issues in finance. Alternatively, Lévy processes can be coupled on the basis of Lévy measures by Lévy copula functions. However, this approach is not necessary in our case.

the dependency, nothing less, nothing more. That is, only when we put the two pieces of information together we have sufficient knowledge about the pair of random variables X, Y.

Assuming that the marginal distribution functions of the random variables are already known, the only further thing we need to know to model the overall evolution is an appropriate copula function. With some simplification, we can distinguish copulas in the form of elliptical distributions and copulas from the Archimedean family. The main difference between these two forms lies in the methods of construction and estimation. While for the latter the primary assumption is to define the generator function, for the former knowledge of the related joint distribution function (e.g. Gaussian, Student) is sufficient.

4.3 Parameter Estimation

There are three main approaches to parameter estimation for copula function-based dependency modeling: the exact maximum likelihood method (EMLM), inference for margins (IFM), and canonical maximum likelihood (CML). For the first-mentioned, all the parameters are estimated in one step, which can be very time consuming, especially for high-dimensional problems or complicated marginal distributions, while the latter two methods are based on estimating the parameters for the marginal distribution and the parameters for the copula function separately. In the IFM approach, the marginal distributions are estimated in the first step and the copula function in the second, whereas for CML empirical distributions are used instead of parametric margins. For more details see any of the empirically oriented literature, such as Cherubini et al. (2004).

5. Data Description

The data set we consider in this study comprises of daily closing prices of four well established equity indices—the Dow Jones Industrial Average (DJI) from the US market, the FTSE 100 (FTSE) from London (UK), the Nikkei 225 (N225) from Tokyo (Japan), and the Swiss Market Index (SMI) from Switzerland—over the preceding 20 years (January 4, 1991 to December 31, 2010). Since the trading days on particular markets are not always harmonized, we had to interpolate missing data. In this way we get four time series of 4,939 log-returns.

Moreover, for better comparison, the initial value of all the indices is set to one. The evolution of the index values after this normalization is depicted in *Figure 1* in the *Appendix*. It is apparent that except for the gradually decreasing N225 the available data allows us to deal with several distinct periods—the instability of the late 1990s, followed by sharp drops in prices, then a solid rise, and even sharper drops during the recent financial crisis. Such rapid changes are a challenge for any risk estimation model. Note also that despite the observed drops in market prices, an initial investment in any of the three indices would have gone up several times in value.

The basic descriptive statistics of the daily log-returns are shown in *Table 2.* In particular, the minimum and maximum return, the mean (expected value), median and standard deviation of the return, and two higher moments—the skewness and kurtosis—are recorded for each index. While the average return of the N225 over the whole time period is negative (approximately -4.2% per annum), the other indices

Table 2 Basic Descriptive Statistics of Equity Indices (daily log-returns)

Index	Min	Max	Mean	Median	St.dev.	Skewness	Kurtosis
DJI	–8.201%	10.508%	0.029%	0.045%	1.100%	–0.060	12.042
FTSE	–9.265%	9.384%	0.019%	0.042%	1.136%	–0.103	9.983
N225	–12.111%	10.086%	–0.017%	0.004%	1.479%	–0.248	7.894
SMI	–8.383%	10.788%	0.032%	0.082%	1.177%	–0.127	9.420

exhibited considerably better performance (4.7% p.a. for the FTSE and more than 7% p.a. for both the DJI and the SMI). Surprisingly, the ranking of the standard deviations (a basic risk measure) is inversely related to that of the average returns—the N225 is highest and the DJI is lowest.

Concerning the minimum and maximum observed returns over one day, the spread may be related to the standard deviation—since the DJI, the FTSE, and the SMI have a similar level of risk, as given by the standard deviations of their returns, the distance of the daily min/max returns is also similar (18.5%). However, for the N225 with its higher risk we recorded a distance of 22%. Next, the skewness of the daily returns is very low in absolute terms. This is true even for the N225 with $skew$ = -0.248. Although the observed skewness might indicate a Gaussian distribution of the daily log-returns of the DJI, after checking the fourth moment—the kurtosis—the assumption of Gaussianity must be clearly rejected. It also seems that extreme returns are recorded more for the DJI ($kurt$ =12) or the FTSE and the SMI than for the N225 ($kurt$ = 7.9).

The indices are denominated in four distinct currencies, namely, the US dollar (USD), the British pound (GBP), the Japanese yen (JPY), and the Swiss franc (CHF). This fact extends our data set to eight distinct time series. As a reference currency the Czech koruna (CZK) is chosen. This implies that there is no riskless investment opportunity. However, it can be assumed that investment in European equities will imply a lower FX rate risk than investment in equities in Japan or the USA due to tighter links between the economies concerned.

The average "return" on FX rates obviously depends on the chosen numeraire (reference currency). In our case, several periods of depreciation were recorded for the Czech koruna during the 1990s. These were followed by a relatively long and stable appreciation, ending in several sudden fluctuations during the last two years connected with both positive and negative returns (see *Figure 2* in the *Appendix*). When averaging over the whole time period, we get a more (GBP, USD) or less (CHF, JPY) negative average daily log-return (long-term appreciation) with a significantly lower standard deviation (as compared to the equity indices)—see *Table 3*. Since from the global point of view the Czech economy is usually regarded as transitional and risky, the maximum returns are much minimal instead negative—sudden depreciations are a common response to shocks and "bad" news, and are usually followed by moderate appreciation. This also implies strongly positive skewness and even higher kurtosis (as compared to the equity indices).

In the investment and risk management process it is no less important to monitor the dependency between particular risk sources—in our case equity index returns and FX rates. Although the Pearson linear measure of dependency is not

Table 3 Basic Descriptive Statistics of FX rates (daily log-returns)

Index	Min	Max	Mean	Median	St.dev.	Skewness	Kurtosis
DJI	–8.201%	10.508%	0.029%	0.045%	1.100%	–0.060	12.042
FTSE	–9.265%	9.384%	0.019%	0.042%	1.136%	–0.103	9.983
N225	–12.111%	10.086%	–0.017%	0.004%	1.479%	–0.248	7.894
SMI	–8.383%	10.788%	0.032%	0.082%	1.177%	–0.127	9.420

an ideal measure of dependency or association between random variables, it can still provide a very good proxy. Briefly, while the dependency between the N225 and any other index is moderate to very low (0.47 to 0.13), the dependency of the FTSE and the SMI is significant (0.76). Moreover, the linear correlation measures of the FTSE and the SMI are very similar with respect to both the DJI and the N225. By contrast, the dependency of FX rates is apparently different, as is implied by the "Czech" point of view—in all cases, the dependency can be regarded as moderate (0.32 to 0.58). Finally, concerning the dependency between equities and FX rates there is apparently no correlation with GBP and USD, while JPY and CHF exhibit a slight negative dependency—see *Table 3* and the relatively low depreciation of these two currencies.

6. Results

In order to examine the power of the suggested model—the marginals defined in terms of the NIG model coupled together by ordinary symmetric copula functions —we will assume (i) Brownian motion and/or the NIG model for the marginal distribution of a single position in equity indices and/or FX rates from the CZK point of view; (ii) a zero-value position of two assets, i.e., suitable combinations of long and short positions, (1;-1) and (-1; 1), under the Gaussian or Student copula function; (iii) a complex equally weighted and "tangential" portfolio in CZK. For comparison purposes, we also provide an estimation of the risk parameters for the latter portfolios on an ex post basis.

The power of the model for estimating risk will be assessed by a backtesting procedure applied to VaR on a daily basis calculated for several significance levels: $q = \{0.0003, 0.005, 0.01, 0.05, 0.15, 0.5\}$. While the first can be connected to internal capital management and a target rating (say, AA), the others are implied by Basel II/ /Solvency II. Finally, $q = 0.5$ allows us to check the estimation for the median.

In order to estimate the parameters of the model, various time spans, τ, will be assumed, with $\tau_{\max}$= 2,000 days, i.e., approximately 8 years of daily data. Since the length of the data is 20 years, we can apply the backtesting procedure on a rolling basis for almost 12 years (N = 2,939 log-returns in particular). The power of the model will be assessed by comparing the assumed number of exceptions with the observation (recall that an exception is given by $I_{L>\mathrm{VaR}(\alpha)}$). For particular qs we therefore get the following number of assumed exceptions: {0.89, 14.7, 29.39, 146.95, 440.85, 1469.5}.

6.1 Ex post Fitting

Before we proceed to the backtesting procedure, i.e., the estimation of the risk measure ex ante with subsequent comparison with the true observations, we evaluate the ex post fitting procedure for two distinct portfolios—an equally weighted one and a tangential one, i.e., we set the weights of all returns as 0.25 for the former and as 0.3DJI&0.7SMI for the latter.[11] Hence, we fit a given model to the historical data, estimate its parameters, and run the Monte Carlo simulation (500,000 independent trials) to get the probability distribution of the returns, for which several descriptive statistics will be calculated—in line with *Tables 2* and *3*, we calculate the mean, median, standard deviation, skewness, kurtosis, and VaR for α = 95%, 99%, and 99.9%.

Since the quality of the ex post modeling can be related to various periods in time, we consider three different time intervals—the whole period, ten years (i.e., approximately 1991 to 2000 and 2001 to 2010), and five years (i.e., approximately 1991 to 1995, 1996 to 2000, 2001 to 2005, and 2006 to 2010). The results we want to get, i.e., historical observations, are recorded in *Tables 7* and *8* in the *Appendix.* It is apparent that the results for the two portfolios are relatively close. Obviously, since the equally weighted portfolio is generally less efficient than the tangential portfolio, the risk-return trade-off will be less pleasant. Next, we observe that during the first decade the portfolios generated higher returns with less volatility, but the returns were more positively skewed with higher peaks. Concerning the five-year periods, the last one leads to the lowest return but the highest volatility and kurtosis, while the preceding one is the most "normal".

Knowing the empirical characteristics, we can try to obtain them by simulating the marginal distribution of particular risk factors and subsequently joining them using the copula function. It seems that the most promising model is NIG with the Student copula function via the IFM approach to parameter estimation—we therefore report these results in tables (see *Tables 9* and *10* in the *Appendix*). It seems however, that the observations of the portfolio returns exhibit quite different skewness and sometimes also kurtosis than desired. Note, however, that the VaR levels seem to be estimated in a much better way.

6.2 Single Position Backtesting

We will now evaluate particular models for single positions—four FX rates from the CZK point of view, four indices in local currencies, and also the indices after recalculation into CZK; however, the dependency of the equity indices and FX rates will be ignored (recall that it is very low). Therefore, we are interested only in the marginal distributions.

The standard way of estimating risk used to be to suppose that the (log)returns of financial quantities follow a Gaussian distribution (a symmetric probability distribution with two parameters). In this case, the two parameter estimation approaches—the method of moments and the maximum likelihood method—provide an equivalent result. A common approach is to use a one-year window for parameter estimation. However, the quite high kurtosis of the daily log-returns suggests that the Gaussian distribution is not a good candidate for modeling the marginals. Indeed, if q is 0.005 or 0.001 the observed number of exceptions is about two or three times higher than the assumption, and assuming q = 0.0003 it is a full ten times higher.

[11] The weights were fixed over the whole period. Since we are examining particular subperiods, the tangential portfolio is (more or less) inefficient. Note, however, that the tangential portfolio consists of the only two indices, which were indicated in *Table 2* as being efficient in the mean-variance sense.

Obviously, there are important differences in the results between particular assets. One can assume that the Gaussian distribution might work better for the indices than for the currencies, since higher kurtosis was documented for the latter. However, we should also take into account the skewness—the risk is measured for the left tail, but the FX rate returns are significantly right-skewed. Therefore, the best results are obtained for JPY with quite high (and positive) skewness and low kurtosis. Among the equity indices, the best Gaussian distribution results are obtained for the DJI, probably through insignificant empirical skewness.

The Gaussian distribution might be accepted as a valid model for risk estimation of single positions for $q = 0.05$ only. Next, the median, $q = 0.5$, is also fitted well—the error is 1% or 2%, i.e., about 20 observations. By contrast, the VaR at the significance level of 0.15 is overestimated—the observed number of exceptions is 15–20% lower. We also tried to increase or decrease the time span, but we did not observe any significant impact. It is therefore clear that the Gaussian distribution should not be used to model the risk of single positions.

Considering the NIG model, the situation is more challenging since there are many ways of defining and estimating the model parameters. In theory, the maximum likelihood method (NIG_{mlm}) should be preferred over the method of moments (NIG_{mm}). Indeed, when we compared the results for a given τ, NIG_{mlm} worked better. Although the results are better than those for the Gaussian distribution, there are still significant deficiencies, especially for lower q. This is because the parameters of the distribution are of different memory. While the standard deviation describes the short-term variability, the fourth moment is related to the heaviness of the tails, i.e., the observations of rare events—and rare events are by definition rare, i.e., can only be observed over a long horizon. We therefore decided to separate the time span used to estimate the mean and variance from the time span used to estimate the kurtosis, while leaving the skewness unsolved. Unfortunately, this approach significantly complicates the application of NIG_{mlm}. Consequently, we proceed further only with NIG_{mm}.

In *Table 4* we review the results obtained by applying the NIG model assuming the time span τ = (mean, variance, skewness, kurtosis) as follows:[12] (60, 60, 60, 2,000), (250, 250, 250, 2,000), and (250, 250, 2,000, 2,000). We denote in bold the results that are the nearest to the assumption, and also provide the acceptance interval according to the Kupiec test for *p*-value 0.1.

We can see that the two time spans for skewness, $\tau = 250$ and $\tau = 2{,}000$, give us similar results, although the former is slightly better. A time span of 60 days works well only for the median; however, the differences between all the approaches are not very significant. When comparing the observed number of exceptions with the assumption, we can regard the NIG_{mm} model as very good for all tail VaR levels. Similarly to the Gaussian assumption, the NIG model also provides better results for FX rates in the tails. By contrast, the risk is underestimated for $q = 15\%$. Since we reported no positive dependency between the FX rates and the indices, it is also natural that the model works well for equity positions in CZK even when no dependency is captured by the model.

[12] We report only the results which we consider to be the most important and interesting. Other results (combinations of τ) are available upon request.

Table 4 Number of Exceptions over 1994–2009 (NIG$_{mm}$)
In particular cells we provide the results obtained for the time span τ (mean, variance, skewness, kurtosis) as follows: (60, 60, 60, 2,000), (250, 250, 250, 2,000), and (250, 250, 2,000, 2,000).

Significance	0.0003	0.005	0.01	0.05	0.15	0.50
Assumption	*0.9*	*14.7*	*29.4*	*147.0*	*440.9*	*1469.5*
Acceptance interval (p = 0.1)	(0,2.8)	(8.9,21.4)	(20.9,38.7)	(127.9,166.8)	(409.3,473.0)	(1322,1616)
DJI	**1**/**1**/2	**14**/11/17	**29**/**29**/32	197/188/**187**	564/503/**502**	1481/1458/**1476**
FTSE	0/0/0	23/22/**21**	42/**37**/**37**	185/**162**/166	521/479/**478**	**1475**/1450/1446
N225	5/**3**/**3**	24/**20**/23	46/**33**/36	178/**147**/157	509/**461**/467	1486/1483/**1458**
SMI	3/**1**/**1**	**16**/18/22	37/**35**/**35**	193/**157**/158	558/**498**/500	**1491**/1445/1445
USD	**0**/3/**0**	12/**14**/9	24/**33**/34	190/**189**/200	595/572/**566**	**1463**/1490/1437
GBP	6/**1**/**1**	**14**/8/8	**28**/23/26	217/**182**/201	605/590/**583**	**1476**/1485/1452
JPY	0/**1**/0	**14**/**14**/10	**27**/25/22	180/**167**/178	556/525/**518**	1473/**1468**/1452
CHF	2/**1**/**1**	9/13/**14**	23/23/**27**	206/**187**/197	655/617/**601**	1454/**1473**/1439
DJI&USD	**1**/**1**/**1**	**18**/19/20	35/35/**34**	172/**164**/168	527/**475**/477	1495/**1473**/**1466**
FTSE&GBP	0/**1**/**1**	21/12/**17**	32/32/32	192/**172**/173	499/**470**/472	**1468**/1439/1443
N225&JPY	4/**1**/**1**	19/**14**/19	30/**28**/32	**142**/132/141	**451**/414/416	**1469**/1486/1456
SMI&CHF	2/**1**/**1**	**11**/10/10	**22**/25/27	156/**139**/137	489/**452**/456	**1469**/1446/1444

6.3 Zero Value Portfolio of Two Assets

After evaluating the single position models, we can proceed to simple dependency modeling—i.e., only two positions are assumed, one long and the second short. This time, however, we use only one time span, τ = (250, 250, 250, 2,000). Instead, we compare the NIG model under the Gaussian (NIG-G) and Student copula (NIG-St) with Brownian motion under the same copulas (BM-G, BM-St). Although we have made it clear that the Gaussian distribution (i.e., Brownian motion) is not a suitable candidate for risk modeling of single positions, it might be that using the Student copula for the dependency between them yields better results. Again, the different time spans for the estimation of kurtosis allow us to apply the method of moments only. By contrast, the parameters of the copula function are estimated on the basis of canonical maximum likelihood for τ = 250 as well (we assume a short memory for the dependency again).

The results, in *Table 5*, clearly document that for modeling the risk of left tails the NIG-St model is the best one, while with q = 0.15 the Gaussian copula should be preferred. In general, the difference between the assumed and observed number of exceptions is very small. There is also no significant effect of changing the positions, i.e., replacing (1;-1) by (-1;1), except for replacing (DJI;SMI) by (SMI;DJI), which is probably implied by the specific pattern of the tail dependency.

6.4 Overall Portfolio Backtesting

Finally, we proceed to risk modeling of the two portfolios—equally weighted and tangential—and its subsequent backtesting. Generally, we assume the following time span: τ = (250, 250, 250, 2,000). However, in the case of the NIG$_{cml}$ approach we also consider τ = (250, 250, 2,000, 2,000) and (60, 60, 60, 2,000)—see *Table 6*.

Table 5 Number of Exceptions for Zero Value Portfolio (Long; Short) over 1994–2009
In particular cells we provide the results obtained for BM-G/BM-St/NIG-G/NIG-St with the time span τ (250, 250, 250, 2,000).

Significance	**0.0003**	**0.005**	**0.01**	**0.05**	**0.15**	**0.50**
Assumption	*0.89*	*14.70*	*29.39*	*146.95*	*440.85*	*1469.50*
Acceptance interval (p = 0.1)	(0,2.8)	(8.9,21.4)	(20.9,38.7)	(127.9,166.8)	(409.3,473.0)	(1322,1616)
DJI&FTSE	7/4/3/**2**	40/24/**17**/12	53/46/38/**33**	139/**143**/139/155	382/410/416/**459**	1491/1491/1492/1491
DJI&N225	5/3/**0**/**0**	31/26/19/**18**	45/41/39/**33**	138/**140**/155/158	389/404/**429**/460	1493/1495/**1480**/**1480**
DJI&SMI	12/3/2/**1**	33/23/19/**16**	43/37/30/**29**	127/127/131/**146**	393/419/**437**/476	1478/1479/**1475**/1476
FTSE&DJI	6/4/0/**0**	26/19/17/**16**	39/36/33/**30**	141/143/**150**/153	424/430/**450**/465	1479/1478/1466/**1467**
N225&DJI	10/3/0/**0**	30/21/**15**/14	44/36/**25**/23	160/163/**155**/168	401/421/**432**/470	1460/**1464**/1461/1462
SMI&DJI	16/8/4/**1**	47/42/31/**27**	60/52/47/**46**	**148**/152/142/155	368/400/405/**447**	1446/**1448**/1447/1447
FTSE&N225	13/6/3/**2**	38/29/22/**18**	54/50/40/**36**	**148**/149/**148**/153	379/398/**430**/453	1446/1444/1459/**1460**
FTSE&SMI	8/5/4/**2**	34/27/18/**15**	48/43/33/**28**	**148**/**148**/143/150	394/409/**435**/459	1461/1461/1473/**1472**
N225&FTSE	9/5/**2**/**2**	31/29/21/**18**	44/42/33/**30**	155/158/**152**/163	387/405/**435**/451	1446/1445/**1461**/1459
SMI&FTSE	12/3/**0**/**0**	37/26/21/**17**	47/43/38/**33**	142/**145**/**145**/155	387/410/**429**/472	1461/1460/**1464**/1462
N225&SMI	8/2/**1**/0	26/**15**/11/10	37/**30**/24/23	140/141/141/**147**	376/404/413/**459**	1477/**1475**/1477/1476
SMI&N225	5/3/**0**/**0**	27/19/**14**/12	44/41/34/**30**	135/136/141/**148**	397/414/**440**/467	1493/1496/**1478**/1480

Table 6 Number of Exceptions over 1994–2009 for Equally Weighted/Tangential Portfolio

Significance	**0.0003**	**0.005**	**0.01**	**0.05**	**0.15**	**0.50**
Assumption	*0.89*	*14.70*	*29.39*	*146.95*	*440.85*	*1469.50*
Acceptance interval (p = 0.1)	(0,2.8)	(8.9,21.4)	(20.9,38.7)	(127.9,166.8)	(409.3,473.0)	(1322,1616)
BM-G (cml)	4/6	26/24	42/49	172/165	419/427	1440/1463
BM-St (cml)	10/10	33/34	51/57	171/166	401/412	1439/1460
NIG-G (ifm)	1/0	25/17	33/38	178/174	446/451	1434/1459
NIG-G (cml)	1/**0**/2 **0**/**0**/1	25/**22**/**22** **17**/**17**/18	34/**32**/45 36/**32**/39	179/**174**/190 172/**170**/190	445/**444**/489 **451**/452/495	1435/1446/**1459** 1456/1457/**1460**
NIG-St (ifm)	0/0	18/17	28/29	185/181	483/487	1432/1455
NIG-St (cml)	0/0/0 **0**/**0**/1	20/17/**15** **15**/16/14	28/**29**/39 **31**/27/36	182/**178**/201 177/**175**/203	**466**/467/517 **470**/**470**/520	1433/1444/**1460** 1456/**1461**/1459

First, concerning the Gaussian distribution, the risk estimation is reliable only for q = 0.15 and 0.5. Otherwise, the number of exceptions indicates that the model is unacceptable. Moreover, there are no particular differences between the two portfolios and, quite surprisingly, the Student copula model is outperformed by the Gaussian copula model.

More promising results are obtained when the NIG model is applied. In particular, it is generally acceptable in almost all the cases considered here. Although all the approaches, i.e., the CML and IFM approaches, the Gaussian and Student copulas, and the various time spans, look very similar, it can be concluded that NIG-St via IFM is the best. Moreover, the results obtained for the tangential portfolio are slightly better; obviously, this portfolio consists of only two assets, so the dependency modeling is not so difficult.

7. Conclusion

FX rate risk modeling and management is a challenging task for financial institutions' risk units. In this paper, we extended the previous analysis of other authors and focused first of all on the performance of the NIG model coupled by either the Gaussian or Student copula function for the case of international equity positions.

In particular, we assumed four distinct stock indexes worldwide and their evolution over the last 20 years. Since the reference currency was the Czech koruna, we also had to deal with four FX rates, which allowed us to evaluate four distinct tasks: (i) single position modeling; (ii) a zero-value position of two assets; (iii) complex equally weighted and "tangential" portfolios evaluated ex-post; and (iv) complex equally weighted and "tangential" portfolios evaluated ex-ante.

Although we identified several cases where the simplifying Gaussian distribution works well (although this was not the case with far left tails), a general recommendation is to adopt the NIG model with the Student copula via the method of moments, since in this case one can freely combine various time spans for parameter estimation. From the selection of combinations we compared, we can recommend a very long time span for the estimation of kurtosis (four years) and a standard window of one year for the mean and variance and also for the dependency. However, concerning the skewness, we did not observe any apparent differences when various time spans were considered, and since the assumed copula function was symmetric, i.e., it does not allow us to fit the asymmetry in the dependency, we would prefer for skewness estimation the same time span as the one used for the copula function estimation.

APPENDIX

Figure 1 Evolution of Equity Indices in Time After Normalization

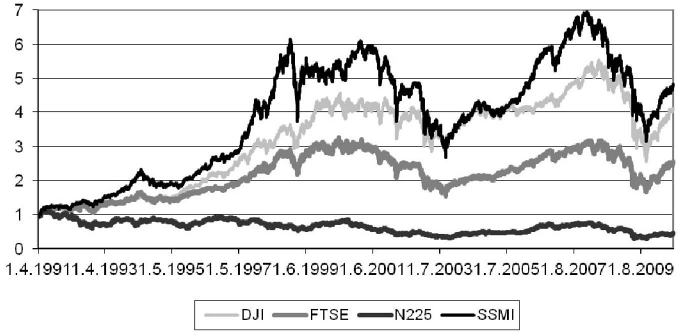

Figure 2 Evolution of FX Rates in Time After Normalization

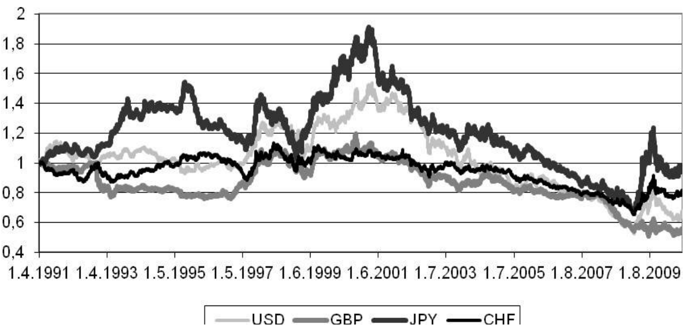

Table 7 Descriptive Statistics of *Equally Weighted* Portfolio

Parameter	1991––2010	1991––2000	2001––2010	1991––1995	1996––2000	2001––2005	2006––2010
mean	0.014%	0.057%	-0.025%	0.041%	0.057%	0.017%	-0.024%
median	0.036%	0.071%	0.009%	0.038%	0.080%	0.044%	0.011%
st.dev.	0.973%	0.853%	1.068%	0.670%	1.019%	1.026%	1.140%
skewness	0.139	0.351	0.078	0.001	0.364	0.182	0.139
kurtosis	8.493	9.279	7.679	7.622	7.255	5.919	10.556
VaR (0.95)	1.537%	1.247%	1.762%	0.979%	1.585%	1.688%	1.840%
VaR (0.99)	2.744%	2.163%	2.962%	1.566%	2.628%	2.756%	3.232%
VaR (0.999)	4.812%	3.719%	5.012%	3.219%	3.766%	3.858%	6.069%

Table 8 Descriptive Statistics of *Tangencial* Portfolio

Parameter	1991– –2010	1991– –2000	2001– –2010	1991– –1995	1996– –2000	2001– –2005	2006– –2010
mean	0.03%	0.08%	-0.02%	0.07%	0.08%	-0.02%	-0.02%
median	0.05%	0.09%	0.00%	0.09%	0.08%	0.00%	0.04%
st.dev.	1.06%	0.94%	1.16%	0.73%	1.09%	1.15%	1.22%
skewness	0.084	-0.037	0.180	-0.384	0.104	0.165	0.188
kurtosis	8.051	7.968	7.687	10.426294	6.017	6.623	8.607
VaR (0.95)	1.63%	1.37%	1.88%	1.06%	1.65%	1.95%	1.89%
VaR (0.99)	3.04%	2.40%	3.37%	1.91%	2.68%	3.33%	3.87%
VaR (0.999)	4.90%	5.00%	4.93%	4.11%	5.03%	4.78%	5.04%

Table 9 Estimation for *Equally Weighted* Portfolio

Parameter	1991– –2010	1991– –2000	2001– –2010	1991– –1995	1996– –2000	2001– –2005	2006– –2010
mean	0.015%	0.059%	-0.025%	0.042%	0.058%	0.017%	-0.022%
median	0.014%	0.052%	-0.021%	0.038%	0.051%	0.014%	-0.023%
st.dev.	0.970%	0.810%	1.089%	0.655%	0.993%	1.022%	1.150%
skewness	0.021	0.151	-0.057	0.052	0.124	0.060	-0.018
kurtosis	9.310	7.956	8.428	5.387	6.891	6.508	11.244
VaR (0.95)	1.467%	1.186%	1.708%	0.992%	1.491%	1.584%	1.751%
VaR (0.99)	2.664%	2.088%	3.037%	1.650%	2.523%	2.666%	3.228%
VaR (0.999)	4.878%	3.630%	5.475%	2.693%	4.321%	4.559%	6.034%

Table 10 Estimation for *Tangencial* Portfolio

Parameter	1991– –2010	1991– –2000	2001– –2010	1991– –1995	1996– –2000	2001– –2005	2006– –2010
mean	0.028%	0.078%	-0.014%	0.069%	0.077%	-0.022%	-0.016%
median	0.027%	0.090%	-0.022%	0.092%	0.084%	-0.025%	-0.030%
st.dev.	1.067%	0.915%	1.180%	0.725%	1.097%	1.149%	1.264%
skewness	0.048	-0.169	0.115	-0.527	-0.005	0.024	0.156
kurtosis	10.790	10.375	9.480	9.657	7.769	6.487	11.188
VaR (0.95)	1.585%	1.336%	1.805%	1.072%	1.647%	1.823%	1.889%
VaR (0.99)	2.992%	2.555%	3.255%	2.087%	2.916%	3.095%	3.528%
VaR (0.999)	5.657%	4.852%	5.852%	3.994%	5.129%	5.223%	6.563%

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